

# HECKE OPERATORS ON HALF-INTEGRAL WEIGHT SIEGEL EISENSTEIN SERIES

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**ABSTRACT.** We construct a basis for the space of half-integral weight Siegel Eisenstein series of level  $4\mathcal{N}$  where  $\mathcal{N}$  is odd and square-free. Then we restrict our attention to those Eisenstein series generated from elements of  $\Gamma_0(4)$ , commenting on why this restriction is necessary for our methods. We directly apply to these forms all Hecke operators attached to odd primes, and we realize the images explicitly as linear combinations of Siegel Eisenstein series. Using this information, we diagonalize the subspace of Eisenstein series generated from elements of  $\Gamma_0(4)$ , obtaining a multiplicity-one result.

## 1. INTRODUCTION

In a seminal paper [5], Shimura established a beautiful correspondence between (Siegel degree 1) cusp forms of half-integral weight  $k/2$ , level  $4\mathcal{N}$  and character  $\chi$ , and elliptic modular forms of integral weight  $k-1$ , level  $2\mathcal{N}$  and character  $\chi^2$ . Essentially the correspondence is established by comparing Hecke-eigenvalues (and using Weil's Converse Theorem to show that the integral weight forms constructed are indeed a modular forms).

In recent work [7] we constructed a basis for the space of degree  $n$ , integral weight  $k$ , arbitrary level  $\mathcal{N}$  and character  $\chi$  Siegel Eisenstein series, and through direct computations we produced a basis of simultaneous eigenforms for the Hecke operators

$$\{T(p), T_j(p^2) : 1 \leq j \leq n, p \text{ prime}, p \nmid \mathcal{N}\};$$

when  $\mathcal{N}$  is square-free, the elements of this basis are also eigenforms for

$$\{T(q), T_j(q^2) : 1 \leq j \leq n, q \text{ prime}, q \mid \mathcal{N}\}$$

and these basis elements are distinguished by their eigenvalues.

Here we extend this work to consider half-integral weight Siegel Eisenstein series. There are several difficulties that arise, since we need to work with automorphy factors. In principle we could work in a cover of the symplectic group, but following Shimura, for any matrix in the congruence subgroup  $\Gamma_0(4)$  (defined below) we make a specific choice for an automorphy factor given by a quotient of Siegel theta series (also defined below). Our computations take advantage of nice properties of these theta series and of generalized Gauss sums (Proposition 2.2). Unfortunately, this also limits our detailed evaluation of Hecke operators to those Siegel Eisenstein series

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generated from elements in  $\Gamma_0(4)$ , and we are unable to give a satisfactory evaluation of the action of  $T_j(4)$ .

Like Shimura, we only consider levels  $4\mathcal{N}$ , and here we especially focus on the case of  $\mathcal{N}$  odd and square-free. In degree 1 this is fully justified as there is no one-fold covering group of a subgroup of  $SL_2(\mathbb{Z})$  that properly contains

$$\{[\gamma, \theta(\gamma\tau)/\theta(\tau)] : \gamma \in \Gamma_0(4) \}$$

(see, for example, Corollary 3.7). However, we are not able to prove this for degree  $n > 1$ , although (also in Corollary 3.7) we prove a partial result toward this. The reasoning used to prove Corollary 3.7 is also used to show that, regardless of the choice of automorphy factor for certain  $\gamma \in Sp_n(\mathbb{Z})$  with  $\gamma \notin \Gamma_0(4)$ , the Siegel Eisenstein series generated from  $\gamma$  is 0 (Proposition 3.6). (In Proposition 3.5 we give necessary conditions on  $\chi$  to have a nonzero Eisenstein series.)

For  $\mathcal{N}$  odd and square-free, we show that the subspace of Eisenstein series generated from elements of  $\Gamma_0(4)$  has a basis of simultaneous eigenforms for the Hecke operators

$$\{T_j(q^2) : q \text{ prime}, q|\mathcal{N}\},$$

and these basis elements are distinguished by their eigenvalues. (As in the case degree 1, the half-integral weight Hecke operator  $T(p)$  is 0 for any prime  $p$ ; see, for example, Proposition 2.1 [6]).

When the degree  $n$  is 1, we recover Shimura's correspondence [5]: With  $\sigma = (\mathcal{N}_0, \mathcal{N}_1)$  varying over all multiplicative partitions of  $\mathcal{N}$  (meaning that  $\mathcal{N}_0\mathcal{N}_1 = \mathcal{N}$ ), we have a basis  $\{\tilde{\mathbb{E}}_\sigma\}$  for the space of weight  $k/2$ , level  $4\mathcal{N}$  and character  $\chi$  Eisenstein series generated from elements of  $\Gamma_0(4)$ , and a basis  $\{\mathbb{E}'_\sigma\}$  for the space of weight  $k-1$ , level  $\mathcal{N}$  and character  $\chi^2$  Eisenstein series, so that for every odd prime  $p$ , the  $T_1(p^2)$ -eigenvalue of  $\tilde{\mathbb{E}}_\sigma$  is the  $T(p)$ -eigenvalue of  $\mathbb{E}'_\sigma$ . For  $n > 1$ , such a correspondence is unclear; below we exhibit the eigenvalues for half-integral weight and integral weight Eisenstein series.

We still assume that  $\mathcal{N}$  is odd and square-free. Take  $k', k \in \mathbb{Z}_+$  with  $k$  odd,  $\chi'$  a character modulo  $\mathcal{N}$ , and  $\chi$  a character modulo  $4\mathcal{N}$ . For  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$  a multiplicative partition of  $\mathcal{N}$ , we have corresponding Eisenstein series  $\mathbb{E}'_\sigma$  and  $\tilde{\mathbb{E}}_\sigma$  of weights  $k'$  and  $k/2$ , levels  $\mathcal{N}$  and  $4\mathcal{N}$ , characters  $\chi'$  and  $\chi$  (respectively). (Note that by Proposition 3.6 [7] and Proposition 4.1, when  $\mathbb{E}'_\sigma \neq 0$  we have  $(\chi'_{\mathcal{N}/\mathcal{N}_0\mathcal{N}_n})^2 = 1$  and when  $\tilde{\mathbb{E}}_\sigma \neq 0$  we have  $(\chi_{4\mathcal{N}/\mathcal{N}_0\mathcal{N}_n})^2 = 1$ .) For a prime  $q|\mathcal{N}$  and  $0 \leq d \leq n$  so that  $q|\mathcal{N}_d$ , by Corollary 4.3 [7] we have

$$\mathbb{E}'_\sigma | T(q) = q^{k'd - d(d+1)/2} \chi'_{\mathcal{N}/q}(\bar{q} X_d M_\sigma, X_d) \mathbb{E}'_\sigma$$

where, for each prime  $q'|\mathcal{N}/q$  and  $0 \leq d' \leq n$  so that  $q'|\mathcal{N}_{d'}$ ,

$$\chi'_{q'}(\bar{q} X_d M_\sigma, X_d) = \begin{cases} \chi'_{q'}(q^{d-d'}) & \text{if } d' \leq d, \\ \chi'_{q'}(q^{d'-d}) & \text{if } d' > d. \end{cases}$$

Also, by Corollary 4.5 [7], we have

$$\begin{aligned} \mathbb{E}'_\sigma | T_j(q^2) &= q^{jd} \sum_{s=0}^j q^{s(2k'-2d+s-j-1)} \chi'_{\mathcal{N}'_0}(q^{2s}) \chi'_{\mathcal{N}'_n}(q^{2(j-s)}) \\ &\quad \cdot \beta_q(d, s) \beta_q(n-d, j-s) \mathbb{E}'_\sigma \end{aligned}$$

where  $\beta_q(m, r)$  is the number of  $r$ -dimensional subspaces of an  $m$ -dimensional space over  $\mathbb{Z}/q\mathbb{Z}$  and  $\mathcal{N}'_i = \mathcal{N}_i/(q, \mathcal{N}_i)$ . In contrast, by Corollary 4.4 we have

$$\begin{aligned} \widetilde{\mathbb{E}}_\sigma | T_j(q^2) &= q^{jd} \sum_{s=0}^j q^{s(k-2d+s-j-1)} \chi_{\mathcal{N}'_0}(q^{2s}) \chi_{\mathcal{N}'_n}(q^{2(j-s)}) \\ &\quad \cdot \beta_q(d, s) \beta_q(n-d, j-s) \widetilde{\mathbb{E}}_\sigma. \end{aligned}$$

In particular,

$$\widetilde{\mathbb{E}}_\sigma | T_n(q^2) = q^{d(k-d-1)} \chi_{\mathcal{N}'_0}(q^{2d}) \chi_{\mathcal{N}'_n}(q^{2(n-d)}).$$

For an odd prime  $p \nmid \mathcal{N}$ , by Corollaries 5.3 and 5.5 [7] we have

$$\mathbb{E}'_\sigma | T(p) = \left( \prod_{0 < d \leq n} \chi'_{\mathcal{N}'_d}(p^d) \right) \prod_{i=1}^n \left( \chi'(p) \overline{\chi'}_{\mathcal{N}'_n}(p^2) p^{k'-i} + 1 \right) \mathbb{E}'_\sigma,$$

$$\begin{aligned} \mathbb{E}'_\sigma | T_j(p^2) &= \beta_p(n, j) \sum_{r+s \leq j} p^{k'(j-r+s)-(j-r)(n+1)} \chi'(p^{j-r+s}) \chi'_{\mathcal{N}'_n}(p^{2(r-s)}) \\ &\quad \cdot \beta_p(j, r) \beta_p(j-r, s) \text{sym}_p(j-r-s) \mathbb{E}'_\sigma \end{aligned}$$

where  $\text{sym}_p(\ell)$  is the number of symmetric,  $\ell \times \ell$ , invertible matrices over  $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ . In contrast, by Theorem 4.5 we have

$$\begin{aligned} \widetilde{\mathbb{E}}_\sigma | T_j(p^2) &= \beta_p(n, j) \sum_{r+s \leq j} p^{k(j-r+s)/2-(j-r)(n+1)} \chi(p^{j-r+s}) \chi_{\mathcal{N}'_n}(p^{2(r-s)}) \\ &\quad \cdot \beta_p(j, r) \beta_p(j-r, s) \left( \frac{\mathcal{G}_1(p)}{\sqrt{p}} \right)^{j-r-s} \text{sym}_p^\psi(j-r-s) \widetilde{\mathbb{E}}_\sigma \end{aligned}$$

where  $\mathcal{G}_1(p)$  is the classical Gauss sum modulo  $p$ ,  $\psi(\ast) = \left( \frac{\ast}{p} \right)$ , and  $\text{sym}_p^\psi(\ell) = \sum_{U \in \mathbb{F}_{\text{sym}}^{\ell, \ell}} \psi(\det U)$ .

Since these eigenvalue of  $\widetilde{\mathbb{E}}_\sigma$  under  $T_j(p^2)$  are not so attractive, in Corollary 4.6 we introduce an alternate set of generators for the local Hecke algebra, obtaining more attractive eigenvalues (similar to what we did in Corollary 5.5 [7]).

As much as possible, we borrow results from [7]. Here the construction of the Siegel Eisenstein series is a bit different because of the automorphy factors involved. As the automorphy factors contain Gauss sums, for the evaluation of the action of the Hecke operators we establish some nice identities between generalized Gauss sums (Propositions 5.1, 5.2, 5.3). Although these identities can surely be established by other methods, we rely on changes of variables to provide elementary arguments.

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## 2. PRELIMINARIES

For  $n \in \mathbb{Z}$ ,  $n > 1$ , Siegel's degree  $n$  upper half-space is defined as

$$\mathfrak{h}_{(n)} = \{X + iY : X, Y \in \mathbb{R}_{\text{sym}}^{n,n}, Y > 0\}$$

where  $Y > 0$  means that, as a quadratic form,  $Y$  is positive definite. The symplectic group  $Sp_n(\mathbb{Z})$  acts on  $\mathfrak{h}_{(n)}$ , where

$$Sp_n(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{2n}(\mathbb{Z}) : A^t B = B^t A, C^t D = D^t C, A^t D - B^t C = I \right\}$$

(here  ${}^t B$  is the transpose of  $B$ ). For  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$  and  $\tau \in \mathfrak{h}_{(n)}$ , the action of  $\gamma$  on  $\tau$  is given by

$$\gamma\tau = (A\tau + B)(C\tau + D)^{-1}.$$

Note that for  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$ ,  $(C \ D)$  is a coprime symmetric pair, meaning that  $C^t D$  is symmetric and for all primes  $p$ ,  $\text{rank}_p(C \ D) = n$  (here  $\text{rank}_p(C \ D)$  denotes the rank of the matrix  $(C \ D)$  modulo  $p$ , meaning we view  $(C \ D)$  as a matrix over  $\mathbb{Z}/p\mathbb{Z}$ ). Conversely, given  $C, D \in \mathbb{Z}^{n,n}$  so that  $(C \ D)$  is a coprime symmetric pair, there is a matrix  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  in  $Sp_n(\mathbb{Z})$ . When  $(C \ D)$  is a pair of integral  $n \times n$  matrices, we write  $(C, D) = 1$  to mean that  $C$  and  $D$  are coprime.

To construct Siegel Eisenstein series of half-integral weight  $k/2$ , we need to make sense of  $(\det(C\tau + D))^{-k/2}$ . Thus we have the following definitions.

**Definition.** An automorphy factor for  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$  is an analytic function  $\varphi_\gamma(\tau)$  on  $\mathfrak{h}_{(n)}$  so that  $|\varphi_\gamma(\tau)|^2 = |\det(C\tau + D)|$ . (Note that it is known that  $\det(C\tau + D) \neq 0$ ; see Proposition 1.2.1 [1].) When we also have  $\gamma' \in Sp_n(\mathbb{Z})$  and  $\varphi_{\gamma'}(\tau)$  an automorphy factor for  $\gamma'$ , we have

$$[\gamma, \varphi_\gamma(\tau)][\gamma', \varphi_{\gamma'}(\tau)] = [\gamma\gamma', \varphi_\gamma(\gamma'\tau)\varphi_{\gamma'}(\tau)].$$

When  $\det D \neq 0$ , we define  $S_{C,D}(\tau)$  by taking

$$\lim_{\lambda \rightarrow 0^+} S_{C,D}(i\lambda I) = \sqrt{\det D} \in \mathbb{R}_+ \cup i\mathbb{R}_+$$

and extending analytically to  $\tau \in \mathfrak{h}_{(n)}$ . Thus with  $\gamma = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$ ,  $\det D \neq 0$ , and  $\varphi_\gamma(\tau)$  an automorphy factor for  $\gamma$ , we have that

$$\frac{\varphi_\gamma(\tau)}{S_{C,D}(\tau)}$$

is analytic with absolute value 1, so  $\varphi_\gamma(\tau) = v(\gamma)S_{C,D}(\tau)$  for some  $v(\gamma)$  with  $|v(\gamma)| = 1$ . We define the basic degree  $n$  Siegel theta series by

$$\theta(\tau) = \sum_{U \in \mathbb{Z}^{1,n}} e\{2^t U U \tau\} \text{ where } e\{*\} = \exp(\pi i \text{Tr}(*)),$$

and for  $(C \ D)$  a coprime symmetric pair with  $\det D \neq 0$ , we define a generalized Gauss sum by

$$\mathcal{G}_C(D) = \sum_{U \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}D} e\{2^t U U D^{-1} C\}.$$

The following result is a special case of Theorem 1.3.13 and Proposition 1.4.5 [1].

**Proposition 2.1.** (*Transformation Formula*) Set

$$\Gamma_0(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : 4|C \right\}.$$

For  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$ , we have

$$\frac{\theta(\gamma\tau)}{\theta(\tau)} = \frac{\overline{\mathcal{G}}_C(D)}{\sqrt{\det D}} S_{C,D}(\tau).$$

Because of this result, we make the following definition.

**Definition.** For  $\gamma \in \Gamma_0(4)$ , we set

$$\tilde{\gamma} = [\gamma, \theta(\gamma\tau)/\theta(\tau)].$$

Note that with  $\gamma, \delta \in \Gamma_0(4)$ , we have  $\tilde{\gamma}\tilde{\delta} = \widetilde{(\gamma\delta)}$ .

The following identities will be useful.

**Proposition 2.2.** Take  $\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$ .

(a) For  $\alpha \in \Gamma_\infty$ , we have  $\theta(\alpha\tau) = \theta(\tau)$ , and for  $Y \in \mathbb{Z}_{\text{sym}}^{n,n}$ , we have

$$\frac{\overline{\mathcal{G}}_C(D + CY)}{\sqrt{\det D + CY}} S_{C,D+CY}(\tau) = \frac{\overline{\mathcal{G}}_C(D)}{\sqrt{\det D}} S_{C,D}(\tau + Y).$$

(b) For  $E \in SL_n(\mathbb{Z})$ , we have  $S_{C,D}(\tau) = S_{EC,ED}(\tau)$ .

(c) For  $E \in GL_n(\mathbb{Z})$ , we have

$$\mathcal{G}_{EC}(ED) = \mathcal{G}_C(D) = \mathcal{G}_{CE}(D^t E^{-1}).$$

*Proof.* (a) For  $\alpha \in \Gamma_\infty$ , we have  $\alpha = \begin{pmatrix} G & GY \\ 0 & {}^t G^{-1} \end{pmatrix}$  for some  $G \in GL_n(\mathbb{Z})$  and  $Y \in \mathbb{Z}_{\text{sym}}^{n,n}$ . Thus for  $U \in \mathbb{Z}^{1,n}$ ,  ${}^t U U G Y {}^t G$  is integral, and  $UG$  varies over  $\mathbb{Z}^{1,n}$  as  $U$  does. Hence

$$\begin{aligned} \theta(\alpha\tau) &= \sum_{U \in \mathbb{Z}^{1,n}} e\{2^t U U G(\tau + Y) {}^t G\} \\ &= \sum_{U \in \mathbb{Z}^{1,n}} e\{2^t (UG)(UG)\tau\} \\ &= \theta(\tau). \end{aligned}$$

So with  $G = I$  and noting that  $\delta\alpha \in \Gamma_0(4)$ , by Proposition 2.1 we have

$$\begin{aligned} \frac{\overline{\mathcal{G}}_C(D + CY)}{\sqrt{\det(D + CY)}} S_{C, D+CY}(\tau) &= \frac{\theta(\delta\alpha\tau)}{\theta(\tau)} \\ &= \frac{\theta(\delta\alpha\tau)}{\theta(\alpha\tau)} \\ &= \frac{\theta(\delta(\tau + Y))}{\theta(\tau + Y)} \\ &= \frac{\overline{\mathcal{G}}_C(D)}{\sqrt{\det D}} S_{C, D}(\tau + Y). \end{aligned}$$

(b) We know that

$$\frac{S_{C, D}(\tau)}{S_{EC, ED}(\tau)}$$

is an analytic function whose square is 1, and whose limit as  $\tau \rightarrow 0$  is also 1. Hence  $S_{C, D}(\tau) = S_{EC, ED}(\tau)$ .

(c) Since  $(ED)^{-1}(EC) = D^{-1}C$  and  $\mathbb{Z}^{1, n}E = \mathbb{Z}^{1, n}$ , we have  $\mathcal{G}_{EC}(ED) = \mathcal{G}_C(D)$ . Also, we have

$$\begin{aligned} \overline{\mathcal{G}}_{CE}(D^t E^{-1}) &= \sum_{U \in \mathbb{Z}^{1, n} / \mathbb{Z}^{1, n} D^t E^{-1}} e\{2^t U U^t E D^{-1} C E\} \\ &= \sum_{U \in \mathbb{Z}^{1, n} / \mathbb{Z}^{1, n} D^t E^{-1}} e\{2(E^t U)(U^t E) D^{-1} C\} \end{aligned}$$

(recall that  $Tr(AB) = Tr(BA)$ .) Take  $U' = U^t E$ . Thus  $U'$  varies over  $\mathbb{Z}^{1, n} E^t / \mathbb{Z}^{1, n} D = \mathbb{Z}^{1, n} / \mathbb{Z}^{1, n} D$  as  $U$  varies over  $\mathbb{Z}^{1, n} / \mathbb{Z}^{1, n} D^t E^{-1}$ . So  $\mathcal{G}_{CE}(D^t E^{-1}) = \mathcal{G}_C(D)$ .  $\square$

We will make use of the following terminology and notation from the theory of quadratic forms. With  $\mathbb{F}$  a field,  $V$  an  $m$ -dimensional  $\mathbb{F}$ -vector space equipped with a quadratic form  $Q$ , and  $A \in \mathbb{F}_{\text{sym}}^{m, m}$ , we write  $V \simeq A$  when  $A$  represents the quadratic form  $Q$  on  $V$  relative to some basis for  $V$ . With  $V \simeq A$ , we say  $V$  is regular if  $\det A \neq 0$ . For a vector  $v \in V$ , we say  $v$  is isotropic if  $Q(v) = 0$ , and anisotropic otherwise. For  $A, A'$  square, symmetric matrices, we sometimes write  $A \perp A'$  for the matrix  $\text{diag}\{A, A'\}$ . With  $a_1, \dots, a_r \in \mathbb{F}$ , we write  $\langle a_1, \dots, a_r \rangle$  for  $\text{diag}\{a_1, \dots, a_r\}$ . We write  $\langle a \rangle^\ell$  to denote the  $\ell \times \ell$  matrix  $\text{diag}\{a, \dots, a\}$ .

For a prime  $q$ , our formulas for the action of the Hecke operators on Eisenstein series will involve the functions we now define. For  $b, c \in \mathbb{Z}$  with  $0 \leq c \leq b$ , set

$$\mu_q(b, c) = \prod_{i=0}^{c-1} (q^{b-i} - 1), \quad \delta_q(b, c) = \prod_{i=0}^{c-1} (q^{b-i} + 1),$$

and set

$$\beta_q(b, c) = \frac{\mu_q(b, c)}{\mu_q(c, c)}.$$

(So  $\beta_q(b, c)$  is the number of  $c$ -dimensional subspaces of a  $b$ -dimensional space over  $\mathbb{Z}/q\mathbb{Z}$ .) We agree that  $\beta_q(0, 0) = 1$  and with  $0 \leq b < c$ ,  $\beta_q(b, c) =$

0. Now fix a character  $\chi$  whose conductor is exactly divisible by  $q$ . Then define

$$\text{sym}_q^\chi(b, c) = \sum_U \chi_q \left( \det \begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix} \right)$$

where  $\mathbb{F} = \mathbb{Z}/q\mathbb{Z}$  and  $\begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix} \in \mathbb{F}_{\text{sym}}^{b+c, b+c}$  with  $\mu$  of size  $b \times b$ . We agree that  $\text{sym}_q^\chi(0, 0) = 1$ , and we set  $\text{sym}_q^\chi(b) = \text{sym}_q^\chi(b, 0)$ .

**Lemma 2.3.** *Let  $q$  be an odd prime, and suppose that  $\chi$  is a character whose conductor is exactly divisible by  $q$ . Then*

$$\text{sym}_q^\chi(b, c) = \begin{cases} \frac{q^{m^2+m-c} \mu(b, b)}{\mu \delta(m-c, m-c)} & \text{if } b+c = 2m \text{ and } \chi_q = 1, \\ \frac{\varepsilon^m q^{m^2} \mu(b, b)}{\mu \delta(m-c, m-c)} & \text{if } b+c = 2m, \chi_q^2 = 1, \text{ and } \chi_q \neq 1, \\ \frac{q^{m^2+m} \mu(b, b)}{\mu \delta(m-c, m-c)} & \text{if } b+c = 2m+1 \text{ and } \chi_q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,  $\text{sym}_q^\chi(b, c) = 0$  unless  $\chi_q^2 = 1$ .

*Proof.* To help us compute  $\text{sym}_q^\chi(b, c)$ , for  $\alpha \in \mathbb{F}^\times$  we let  $\text{sym}_q(b, c; \alpha)$  denote the number of  $U = \begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix} \in \mathbb{F}_{\text{sym}}^{b+c, b+c}$  with  $\mu$   $b \times b$  and  $\det U = \alpha$ .

Set  $r = b+c$ . With  $V$  an  $r$ -dimensional vector space over  $\mathbb{F}$ , an invertible matrix  $A \in \mathbb{F}_{\text{sym}}^{r, r}$  defines a regular quadratic form  $Q$  on  $V$ . Since  $q$  is odd, by Theorem 2.11 [2]  $V$  has a diagonal basis (relative to the quadratic form  $Q$ ). Then by Proposition 2.51 and Theorem 2.52 of [2], we have that  $V \simeq I$  or  $V \simeq I \perp \langle \omega \rangle$  where  $\omega$  is a fixed, non-square element of  $\mathbb{F}^\times$ . If we change the basis for  $V$  by a matrix  $G \in GL_r(\mathbb{F})$ , we get  $V \simeq {}^t G I G$  or  $V \simeq {}^t G (I \perp \langle \omega \rangle) G$ . Thus when  $V \simeq I$ , any matrix for the quadratic form on  $V$  has determinant  $\alpha^2$  for some  $\alpha \in \mathbb{F}^\times$ , and when  $V \simeq I \perp \langle \omega \rangle$ , any matrix for the quadratic form on  $V$  has determinant  $\alpha^2 \omega$  for some  $\alpha \in \mathbb{F}^\times$ . Also, note that for  $\alpha \in \mathbb{F}^\times$ ,

$$\begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix} \mapsto \begin{pmatrix} I & \\ & \alpha \end{pmatrix} \begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix} \begin{pmatrix} I & \\ & \alpha \end{pmatrix}$$

gives us a bijection between the matrices counted by  $\text{sym}(b, c; 1)$  and those counted by  $\text{sym}(b, c; \alpha^2)$ , and between the matrices counted by  $\text{sym}(b, c; \omega)$  and those counted by  $\text{sym}(b, c; \omega \alpha^2)$ . Thus

$$\text{sym}_q^\chi(b, c) = \frac{1}{2} (\text{sym}_q(b, c; 1) + \chi_q(\omega) \text{sym}_q(b, c; \omega)) \sum_{\alpha \in \mathbb{F}^\times} \chi_q(\alpha^2).$$

(The factor of  $1/2$  is to account for the value  $\alpha^2$  appearing twice as  $\alpha$  varies over  $\mathbb{F}^\times$ .) Thus  $\text{sym}_q^\chi(b, c) = 0$  if  $\chi_q^2 \neq 1$ .

On the other hand, we can compute

$$\sum_{\alpha^2 \neq 0} \text{sym}_q(b, c; \alpha^2) \text{ and } \sum_{\alpha^2 \neq 0} \text{sym}_q(b, c; \omega \alpha^2)$$

as follows. First we choose a basis  $y_{b+1}, \dots, y_{b+c}$  for a dimension  $c$  subspace of  $V$  that is totally isotropic (meaning that the quadratic form on  $\mathbb{F}y_{b+1} \oplus$

$\cdots \oplus \mathbb{F}y_{b+c}$  is identically 0). Then we extend this to a basis  $y_1, \dots, y_{b+c}$  for  $V$ . In this way we construct all bases for  $V$  relative to which  $V \simeq \begin{pmatrix} \mu & \nu \\ t_\nu & 0 \end{pmatrix}$  where  $\mu$  is  $b \times b$ . There are  $o(V)$  bases that yield the same matrix, where  $o(V)$  denotes the order of the orthogonal group of  $V$ . Using Theorems 2.19, 2.59, 2.60 from [2] we compute these quantities to obtain the formulas for  $\text{sym}_q^\chi(b, c)$ .  $\square$

### 3. DEFINING EISENSTEIN SERIES

Fix  $\mathcal{N} \in \mathbb{Z}_+$ . The (degree zero) cusps of the Siegel half-space  $\mathfrak{h}_{(n)}$  under the action of the congruence subgroup  $\Gamma_0(4\mathcal{N})$  correspond to the elements of the double quotient  $\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(4\mathcal{N})$  where

$$\Gamma_\infty = \left\{ \begin{pmatrix} G & GY \\ 0 & {}^tG^{-1} \end{pmatrix} : G \in GL_n(\mathbb{Z}), Y \in \mathbb{Z}_{\text{sym}}^{n,n} \right\},$$

$$\Gamma_0(4\mathcal{N}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : 4\mathcal{N} | C \right\}.$$

Given  $\gamma \in Sp_n(\mathbb{Z})$ , we want to construct a half-integral weight Eisenstein series generated by the  $\Gamma_0(4\mathcal{N})$ -orbit of  $\Gamma_\infty \gamma$  and transforming with some character  $\chi$ . We begin by defining an Eisenstein series for the group

$$\Gamma(4\mathcal{N}) = \{ \beta \in Sp_n(\mathbb{Z}) : \beta \equiv I \pmod{4\mathcal{N}} \},$$

as follows. With  $\delta \in Sp_n(\mathbb{Z})$  and  $\varphi_\delta(\tau)$  an automorphy factor for  $\delta$  (assuming  $\varphi_\delta(\tau) = \theta(\delta\tau)/\theta(\tau)$  when  $\delta \in \Gamma_0(4)$ ), we set

$$1(\tau)[[\delta, \varphi_\delta(\tau)]] = (\varphi_\delta(\tau))^{-k}.$$

Then with

$$\Gamma_\infty \Gamma(4\mathcal{N}) = \cup_{\delta^*} \Gamma_\infty \delta^* \text{ (disjoint),}$$

we set

$$\mathbb{E}^*(\tau) = \sum_{\delta^*} 1(\tau) | \widetilde{\delta^*}.$$

Since  $1(\tau) | \widetilde{\beta} = 1$  for any  $\beta \in \Gamma_\infty$ , the (formal) sum for  $\mathbb{E}^*(\tau)$  is well-defined, and provided  $n > (k+1)/2$ , the sum on  $\delta^*$  is absolutely convergent and in fact is analytic (in all the variables of  $\tau$ ). Also,  $\mathbb{E}^* \neq 0$  as  $\lim_{\tau \rightarrow i\infty} \mathbb{E}^*(\tau) = 1$  since  $\lim_{\tau \rightarrow i\infty} 1(\tau) | \widetilde{\delta^*} = 0$  unless  $\delta^* \in \Gamma_\infty$ . For any  $\alpha \in \Gamma(4\mathcal{N})$ , we have  $\Gamma_\infty \Gamma(4\mathcal{N}) = \cup_{\delta^*} \Gamma_\infty \delta^* \alpha$  (disjoint) and  $\widetilde{\delta^* \alpha} = \widetilde{\delta^*} \alpha$ , so

$$\mathbb{E}^*(\tau) | \widetilde{\alpha} = \sum_{\delta^*} 1(\tau) | \widetilde{\delta^*} \alpha = \mathbb{E}^*(\tau).$$

Thus  $\mathbb{E}^*$  is a (nonzero) Eisenstein series for  $\Gamma(4\mathcal{N})$  with weight  $k/2$ .

Now take  $\gamma \in Sp_n(\mathbb{Z})$  with automorphy factor  $\varphi_\gamma(\tau)$ , and fix a character  $\chi$  modulo  $4\mathcal{N}$ . Assume that  $n > (k+1)/2$ . We would like to define an Eisenstein series supported on the  $\Gamma_0(4\mathcal{N})$ -orbit of  $\Gamma_\infty \gamma$  by

$$\sum_{\delta} \overline{\chi}(\delta) \mathbb{E}^*(\tau) | \widetilde{\gamma \delta}$$



where

$$\Gamma_\infty \gamma \Gamma_0(4\mathcal{N}) = \cup_\delta \Gamma_\infty \Gamma(4\mathcal{N}) \gamma \delta \text{ (disjoint) and } \chi(\delta) = \chi(\det D_\delta).$$

However, this sum is not well-defined unless, for all  $\alpha \in \Gamma_0(4\mathcal{N})$  so that  $\Gamma_\infty \Gamma(4\mathcal{N}) \gamma \alpha = \Gamma_\infty \Gamma(4\mathcal{N}) \gamma$ , we have

$$\overline{\chi}(\alpha) \mathbb{E}^*(\tau) | \widetilde{\gamma} \widetilde{\alpha} = \mathbb{E}^*(\tau) | \widetilde{\gamma}.$$

With this in mind, we have the following definitions and lemma.

**Definition.** Fix a level  $4\mathcal{N}$ , a character  $\chi$  module  $4\mathcal{N}$ ,  $\gamma \in Sp_n(\mathbb{Z})$ , and automorphy factor  $\varphi_\gamma(\tau)$  for  $\gamma$  (with  $\varphi_\gamma(\tau) = \theta(\gamma\tau)/\theta(\tau)$  if  $\gamma \in \Gamma_0(4)$ ). Set

$$\Gamma_\gamma = \{ \alpha \in \Gamma_0(4\mathcal{N}) : \Gamma_\infty \Gamma(4\mathcal{N}) \gamma \alpha = \Gamma_\infty \Gamma(4\mathcal{N}) \gamma \},$$

and set

$$\Gamma'_{\gamma, \chi} = \{ \alpha \in \Gamma_\gamma : \overline{\chi}(\alpha) \mathbb{E}^*(\tau) | \widetilde{\gamma} \widetilde{\alpha} = \mathbb{E}^*(\tau) | \widetilde{\gamma} \}.$$

(Note that  $\Gamma_\gamma$  is a group.) We give an alternative definition of  $\Gamma'_{\gamma, \chi}$  as follows.

For  $\alpha \in \Gamma_\gamma$ , we have  $\gamma \alpha \gamma^{-1} \in \Gamma_\infty \Gamma(4\mathcal{N})$  and hence  $\mathbb{E}^*(\tau) | \widetilde{\gamma \alpha \gamma^{-1}} = \mathbb{E}^*(\tau)$ . Thus for  $\alpha \in \Gamma_\gamma$ , we have  $\alpha \in \Gamma'_{\gamma, \chi}$  if and only if

$$\overline{\chi}(\alpha) \mathbb{E}^*(\tau) | \widetilde{\gamma \alpha \gamma^{-1}} = \mathbb{E}^*(\tau) | \widetilde{\gamma \alpha \gamma^{-1}}.$$

Here  $\varphi_{\gamma^{-1}}(\tau) = \frac{1}{\varphi_\gamma(\gamma^{-1}\tau)}$  so that  $\widetilde{\gamma \alpha \gamma^{-1}} = \widetilde{I}$ . So defining  $\zeta_\gamma(\alpha) : \Gamma_\gamma \rightarrow \mathbb{C}^\times$  by the relation

$$\widetilde{\gamma \alpha \gamma^{-1}} (\gamma \alpha^{-1} \gamma^{-1}) = [I, \zeta_\gamma(\alpha)]$$

we have

$$\Gamma'_{\gamma, \chi} = \left\{ \alpha \in \Gamma_\gamma : \chi \zeta_\gamma^k(\alpha) = 1 \right\}.$$

We now establish some basic properties about  $\zeta_\gamma$ .

**Lemma 3.1.** Fix a level  $4\mathcal{N}$ ,  $\gamma \in Sp_n(\mathbb{Z})$ , and  $\varphi_\gamma(\tau)$  an automorphy factor for  $\gamma$ , with  $\varphi_\gamma(\tau) = \theta(\gamma\tau)/\theta(\tau)$  if  $\gamma \in \Gamma_0(4)$ . The map  $\zeta_\gamma : \Gamma_\gamma \rightarrow \mathbb{C}^\times$  is a homomorphism taking values in the multiplicative group  $\{\pm 1, \pm i\}$ . Thus  $\chi \zeta_\gamma^k : \Gamma_\gamma \rightarrow \mathbb{C}^\times$  is a homomorphism with finite image and kernel  $\Gamma'_{\gamma, \chi}$ . If  $\gamma \in \Gamma_0(4)$  and  $\alpha \in \Gamma_\gamma$ , we have  $\zeta_\gamma(\alpha) = 1$ .

*Proof.* Take  $\alpha \in \Gamma_\gamma$ ; note that we have  $\alpha, \gamma \alpha \gamma^{-1} \in \Gamma_0(4)$ . When  $\gamma \in \Gamma_0(4)$ , we have

$$\widetilde{\gamma \alpha \gamma^{-1}} = \widetilde{(\gamma \alpha \gamma^{-1})}$$

and hence  $\zeta_\gamma(\alpha) = 1$ .

Now suppose that  $\gamma \notin \Gamma_0(4)$ . For  $\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$ , let  $\psi_\delta(\tau) = \det(C\tau + D)$ . One easily checks that for  $\delta' \in Sp_n(\mathbb{Z})$ , we have  $\psi_\delta(\delta'\tau) \psi_{\delta'}(\tau) = \psi_{\delta\delta'}(\tau)$ . Also, for any  $\delta \in Sp_n(\mathbb{Z})$  with automorphy factor  $\varphi_\delta(\tau)$  the function  $(\varphi_\delta(\tau))^2 / \psi_\delta(\tau)$  is analytic with absolute value 1, thus for some  $w(\delta)$  with  $|w(\delta)| = 1$ , we have  $(\varphi_\delta(\tau))^2 / \psi_\delta(\tau) = w(\delta)$ . Then we have

$$w(\gamma^{-1}) \psi_{\gamma^{-1}}(\tau) = \frac{1}{w(\gamma) \psi_\gamma(\gamma^{-1}\tau)}$$

and so

$$\frac{1}{w(\gamma)} = w(\gamma^{-1}) \psi_\gamma(\gamma^{-1}\tau) \psi_{\gamma^{-1}}(\tau) = w(\gamma^{-1}).$$

So

$$\begin{aligned}\zeta_\gamma(\alpha)^2 &= w(\gamma) \psi_\gamma(\alpha\gamma^{-1} \cdot \gamma\alpha^{-1}\gamma^{-1}\tau) w(\alpha) \psi_\alpha(\gamma^{-1} \cdot \gamma\alpha^{-1}\gamma^{-1}\tau) \\ &\quad \cdot \frac{\psi_{\gamma^{-1}}(\gamma\alpha^{-1}\gamma^{-1}\tau)}{w(\gamma)} w(\gamma\alpha^{-1}\gamma^{-1}) \psi_{\gamma\alpha^{-1}\gamma^{-1}}(\tau) \\ &= w(\alpha) w(\gamma\alpha^{-1}\gamma^{-1}).\end{aligned}$$

Since  $\alpha \in \Gamma_\gamma \subseteq \Gamma_0(4\mathcal{N})$ , we have  $\gamma\alpha^{-1}\gamma^{-1} \in \Gamma_\infty\Gamma(4\mathcal{N}) \subseteq \Gamma_0(4\mathcal{N})$ , and hence by the Transformation Formula we know that  $w(\alpha), w(\gamma\alpha^{-1}\gamma^{-1})$  are squares of normalised Gauss sums. Thus we have  $\zeta_\gamma(\alpha) \in \{\pm 1, \pm i\}$ .

To see that  $\zeta_\gamma$  is a homomorphism, take  $\alpha, \delta \in \Gamma_\gamma$ . Then, since we have  $\alpha, \delta, \gamma\alpha^{-1}\gamma^{-1}, \gamma\delta^{-1}\gamma^{-1} \in \Gamma_0(4)$ ,

$$\begin{aligned}[I, \zeta_\gamma(\alpha\delta)] &= \widetilde{\gamma\alpha\delta\gamma^{-1}}(\gamma\delta^{-1}\alpha^{-1}\gamma^{-1}) \\ &= (\widetilde{\gamma\alpha\gamma^{-1}})(\widetilde{\gamma\delta\gamma^{-1}})(\widetilde{\gamma\delta^{-1}\gamma^{-1}})(\widetilde{\gamma\alpha^{-1}\gamma^{-1}}) \\ &= (\widetilde{\gamma\alpha\gamma^{-1}})[I, \zeta_\gamma(\delta)](\widetilde{\gamma\alpha^{-1}\gamma^{-1}}) \\ &= (\widetilde{\gamma\alpha\gamma^{-1}})(\widetilde{\gamma\alpha^{-1}\gamma^{-1}})[I, \zeta_\gamma(\delta)] \\ &= [I, \zeta_\alpha(\delta)][I, \zeta_\gamma(\delta)] \\ &= [I, \zeta_\gamma(\alpha)\zeta_\gamma(\delta)].\end{aligned}$$

Hence  $\zeta_\gamma$  is a homomorphism and  $\chi\zeta^k : \Gamma_\gamma \rightarrow \mathbb{C}^\times$  is a homomorphism with finite image. So  $\Gamma'_{\gamma, \chi}$  is a normal subgroup of finite index in  $\Gamma_\gamma$ .  $\square$

Now fix  $\gamma \in Sp_n(\mathbb{Z})$  and fix an automorphy factor  $\varphi_\gamma(\tau)$  for  $\gamma$ ; assume that  $n > (k+1)/2$ . Set

$$\mathbb{E}_\gamma = \frac{1}{[\Gamma_\gamma : \Gamma'_{\gamma, \chi}]} \sum \overline{\chi}(\delta'\delta) \mathbb{E}^*|\widetilde{\gamma\delta'\delta}$$

where

$$\Gamma_0(4\mathcal{N}) = \cup_\delta \Gamma_\gamma \delta \text{ (disjoint), } \Gamma_\gamma = \cup_{\delta'} \Gamma'_{\gamma, \chi} \delta' \text{ (disjoint).}$$

Hence

$$\Gamma_0(4\mathcal{N}) = \cup_{\delta', \delta} \Gamma'_{\gamma, \chi} \delta' \delta \text{ (disjoint).}$$

Since  $\Gamma'_{\gamma, \chi}$  has finite index in  $\Gamma_\gamma$ ,  $\Gamma(4\mathcal{N}) \subseteq \Gamma_\gamma$ , and  $\Gamma(4\mathcal{N})$  has finite index in  $\Gamma_0(4\mathcal{N})$ , we have  $\mathbb{E}_\gamma$  defined as a finite sum. To see that  $\mathbb{E}_\gamma$  is well-defined, take  $\beta \in \Gamma'_{\gamma, \chi}$ . Thus

$$\begin{aligned}\overline{\chi}(\beta) \mathbb{E}^*|\widetilde{\gamma\beta} &= \overline{\chi}(\beta) \mathbb{E}^*|[I, \zeta_\gamma(\beta)]\widetilde{\gamma\beta\gamma^{-1}}|\widetilde{\gamma} \\ &= \mathbb{E}^*|\widetilde{\gamma\beta\gamma^{-1}}|\widetilde{\gamma} \\ &= \mathbb{E}^*|\widetilde{\gamma}\end{aligned}$$

since  $\gamma\beta\gamma^{-1} \in \Gamma_\infty\Gamma(4\mathcal{N})$ . Thus  $\mathbb{E}_\gamma$  is well-defined. With  $\delta, \delta'$  varying as above, by Lemma 3.1 we have

$$\begin{aligned}[\Gamma_\gamma : \Gamma'_{\gamma, \chi}] \mathbb{E}_\gamma &= \sum_{\delta' \delta} \overline{\chi}(\delta'\delta) \mathbb{E}^*|\widetilde{\gamma\delta'\delta} \\ &= \sum_{\delta} \overline{\chi}(\delta) \left( \sum_{\delta'} \overline{\chi\zeta_\gamma^{-k}}(\delta') \right) \mathbb{E}^*|\widetilde{\gamma\delta}.\end{aligned}$$

Since  $\chi\zeta_\gamma^k$  is a homomorphism on  $\Gamma_\gamma$ , we have that  $\mathbb{E}_\gamma = 0$  unless  $\chi\zeta_\gamma^k$  is trivial on  $\Gamma_\gamma$  (meaning that  $\Gamma'_{\gamma,\chi} = \Gamma_\gamma$ ). Note that for any  $\alpha \in \Gamma_0(4\mathcal{N})$ , we have

$$\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_\delta \Gamma_\infty\Gamma(4\mathcal{N})\gamma\delta\alpha \text{ (disjoint),}$$

and since  $\widetilde{\delta\alpha} = \widetilde{\delta}\alpha$  for any  $\delta$  as above, we get

$$\mathbb{E}_\gamma|\widetilde{\alpha} = \sum_\delta \overline{\chi}(\delta)\mathbb{E}^*|\widetilde{\gamma\delta\alpha} = \chi(\alpha)\mathbb{E}_\gamma.$$

Hence  $\mathbb{E}_\gamma$  is a weight  $k/2$ , level  $4\mathcal{N}$  Eisenstein series with character  $\chi$ .

Henceforth, we assume that  $n, k, \mathcal{N} \in \mathbb{Z}_+$  are fixed with  $k$  odd and  $n > (k+1)/2$ , and we fix a character  $\chi$  modulo  $4\mathcal{N}$ .

**Proposition 3.2.** *Take  $\gamma = \begin{pmatrix} * & * \\ M_0 & N_0 \end{pmatrix} \in Sp_n(\mathbb{Z})$ ,  $\varphi_\gamma(\tau)$  an automorphy factor for  $\gamma$ , and  $\chi$  a character modulo  $4\mathcal{N}$ . Suppose that  $\Gamma'_{\gamma,\chi} = \Gamma_\gamma$ . Then  $\mathbb{E}_\gamma \neq 0$  and*

$$\mathbb{E}_\gamma(\tau) = \sum_\beta \overline{\chi}(\beta)1(\tau)|\widetilde{\gamma}\widetilde{\beta}$$

where  $\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_\beta \Gamma_\infty\gamma\beta$  (disjoint). If  $\gamma \in \Gamma_0(4)$  then

$$\mathbb{E}_\gamma(\tau) = \sum_{(M\ N)} \overline{\chi}(M, N) \left( \frac{\overline{\mathcal{G}}_{M(N)}}{\sqrt{\det N}} S_{M,N}(\tau) \right)^{-k}$$

where  $GL_n(\mathbb{Z})(M\ N)$  varies over  $GL_n(\mathbb{Z})(M_0\ N_0)\Gamma_0(4\mathcal{N})$  and  $\chi(M, N) = \chi(\beta)$  where  $\beta \in \Gamma_0(4\mathcal{N})$  so that  $GL_n(\mathbb{Z})(M\ N) = GL_n(\mathbb{Z})(M_0\ N_0)\beta$ .

*Proof.* With  $\delta^* \in \Gamma(4\mathcal{N})$  and  $\delta \in \Gamma_0(4\mathcal{N})$  so that  $\Gamma_\infty\Gamma(4\mathcal{N}) = \cup_{\delta^*} \Gamma_\infty\delta^*$  (disjoint),  $\Gamma_0(4\mathcal{N}) = \cup_\delta \Gamma_\gamma\delta$  (disjoint), one easily sees that  $\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_{\delta^*, \delta} \Gamma_\infty\delta^*\gamma\delta$  (disjoint). Also,  $\gamma^{-1}\delta^*\gamma \in \Gamma(4\mathcal{N})$ , so

$$\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_{\delta^*, \delta} \Gamma_\infty\gamma(\gamma^{-1}\delta^*\gamma)\delta \text{ (disjoint).}$$

Since  $\gamma^{-1}\delta^*\gamma \in \Gamma_\gamma = \Gamma'_{\gamma,\chi}$ , we have

$$1(\tau)|\widetilde{\delta^*\gamma} = 1(\tau)|\widetilde{\gamma}|\widetilde{\gamma^{-1}\delta^*\gamma} = 1(\tau)|\widetilde{\gamma}|\widetilde{\gamma^{-1}\delta^*\gamma}.$$

Hence

$$\mathbb{E}_\gamma(\tau) = \sum_\beta \overline{\chi}(\beta)1(\tau)|\widetilde{\gamma}\widetilde{\beta}$$

where  $\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_\beta \Gamma_\infty\gamma\beta$  (disjoint). Also, for  $\delta^* \in \Gamma_\infty\Gamma(4\mathcal{N})$  and  $\delta \in \Gamma_0(4\mathcal{N})$ , we have

$$\lim_{\tau \rightarrow i\infty} 1(\tau)|\widetilde{\delta^*\gamma}\widetilde{\delta\gamma}^{-1} = 0$$

unless  $\delta^*\gamma\delta\gamma^{-1} \in \Gamma_\infty$ , in which case  $\gamma\delta\gamma^{-1} \in \Gamma_\infty\Gamma(4\mathcal{N})$  and so  $\delta \in \Gamma_\gamma$ . Thus

$$\lim_{\tau \rightarrow i\infty} \mathbb{E}_\gamma(\tau)|\widetilde{\gamma}^{-1} = \lim_{\tau \rightarrow i\infty} \mathbb{E}^*(\tau) = 1,$$

so  $\mathbb{E}_\gamma \neq 0$ .

Now suppose that  $\gamma \in \Gamma_0(4)$ . If  $\beta \in \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in \Gamma_\infty\gamma\Gamma_0(4\mathcal{N})$  then  $(M\ N) \in GL_n(\mathbb{Z})(M_0\ N_0)\Gamma_0(4\mathcal{N})$ . On the other hand, suppose  $(M\ N)$  is a coprime symmetric pair. Thus there is some  $\alpha = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in \Gamma_\infty\gamma\Gamma_0(4\mathcal{N})$ ,

and if  $(M \ N) \in GL_n(\mathbb{Z})(M_0 \ N_0)\delta$  for some  $\delta \in \Gamma_0(4\mathcal{N})$ , then one easily checks that  $\alpha \in \Gamma_\infty\gamma\delta$ . Further, with  $\alpha' = \begin{pmatrix} * & * \\ M' & N' \end{pmatrix}$ , we have

$$GL_n(\mathbb{Z})(M \ N) = GL_n(\mathbb{Z})(M' \ N') = GL_n(\mathbb{Z})(M_0 \ N_0)\delta$$

if and only if  $\Gamma_\infty\alpha = \Gamma_\infty\alpha' = \Gamma_\infty\gamma\delta$ . Thus

$$\mathbb{E}_\gamma(\tau) = \sum_{(M \ N)} \bar{\chi}(M, N) \left( \frac{\bar{\mathcal{G}}_M(N)}{\sqrt{\det N}} S_{M, N}(\tau) \right)^{-k}$$

where  $GL(\mathbb{Z})(M \ N)$  varies (once) over  $GL_n(\mathbb{Z})(M_0 \ N_0)\Gamma_0(4\mathcal{N})$ .  $\square$

Next we describe a basis for the space of Eisenstein series.

**Proposition 3.3.** *Let  $\gamma$  vary over elements of  $Sp_n(\mathbb{Z})$  so that*

$$Sp_n(\mathbb{Z}) = \cup_\gamma \Gamma_\infty\gamma\Gamma_0(4\mathcal{N}).$$

*Then the corresponding nonzero weight  $k/2$ , level  $4\mathcal{N}$  Eisenstein series with character  $\chi$  are linearly independent. Further, suppose that  $\gamma \in Sp_n(\mathbb{Z})$ ,  $\beta \in \Gamma_0(4\mathcal{N})$ ,  $\beta^* \in \Gamma_\infty\Gamma(4\mathcal{N})$  so that  $\alpha = \beta^*\gamma\beta$ ; set  $\tilde{\alpha} = \tilde{\beta}^*\tilde{\gamma}\tilde{\beta}$ . Then  $\mathbb{E}_\alpha = \chi(\beta)\mathbb{E}_\gamma$ .*

*Proof.* Suppose  $\gamma, \beta \in Sp_n(\mathbb{Z})$  so that  $\beta \notin \Gamma_\infty\gamma\Gamma_0(4\mathcal{N})$  and  $\mathbb{E}_\gamma, \mathbb{E}_\beta \neq 0$ . As shown in the proof of Proposition 3.2,

$$\lim_{\tau \rightarrow i\infty} \mathbb{E}_\gamma|\tilde{\gamma}^{-1} = 1,$$

but as

$$\lim_{\tau \rightarrow i\infty} 1(\tau)|\tilde{\delta}^*\tilde{\beta}\tilde{\delta}\tilde{\gamma}^{-1} = 0$$

for all  $\delta^* \in \Gamma_\infty\Gamma(4\mathcal{N})$  and  $\delta \in \Gamma_0(4\mathcal{N})$ , we have

$$\lim_{\tau \rightarrow i\infty} \mathbb{E}_\beta(\tau)|\tilde{\gamma}^{-1} = 0.$$

Consequently  $\mathbb{E}_\gamma$  is linearly independent of the set

$$\{\mathbb{E}_\beta : \beta \in Sp_n(\mathbb{Z}), \beta \notin \Gamma_\infty\gamma\Gamma_0(4\mathcal{N})\}.$$

Now take  $\gamma \in Sp_n(\mathbb{Z})$ ,  $\beta^* \in \Gamma_\infty$ ,  $\beta \in \Gamma_0(4\mathcal{N})$ . Set  $\tilde{\alpha} = \tilde{\beta}^*\tilde{\gamma}\tilde{\beta}$ . With  $\delta'$  varying so that

$$\Gamma_\infty\alpha\Gamma_0(4\mathcal{N}) = \cup_{\delta'} \Gamma_\infty\alpha\delta' \text{ (disjoint),}$$

we have

$$\Gamma_\infty\gamma\Gamma_0(4\mathcal{N}) = \cup_{\delta'} \Gamma_\infty\gamma(\beta\delta') \text{ (disjoint).}$$

Hence

$$\mathbb{E}_\alpha = \sum_{\delta'} \bar{\chi}(\delta')\mathbb{E}^*|\tilde{\alpha}\tilde{\delta}' = \sum_{\delta'} \bar{\chi}(\delta')\mathbb{E}^*|\tilde{\gamma}(\tilde{\beta}\tilde{\delta}') = \chi(\beta)\mathbb{E}_\gamma$$

(recall that for  $\beta, \delta' \in \Gamma_0(4)$ , we have  $\tilde{\beta}\tilde{\delta}' = \tilde{\beta}\tilde{\delta}'$ ).  $\square$

In this paper we are particularly interested in Eisenstein series of level  $4\mathcal{N}$  where  $\mathcal{N}$  is odd and square-free. Below we introduce some terminology and then exhibit a set of representatives for the degree zero cusps.

**Definition.** Suppose  $\mathcal{N} \in \mathbb{Z}_+$  is odd and square-free. We say

$$\sigma = ((\mathcal{N}_0, \dots, \mathcal{N}_n), (d, d', \epsilon))$$

is an admissible type for level  $4\mathcal{N}$  if  $(\mathcal{N}_0, \dots, \mathcal{N}_n)$  is a multiplicative partition of  $\mathcal{N}$ ,  $d, d' \in \mathbb{Z}_{\geq 0}$  so that  $d + d' \leq n$ , and  $\epsilon = +$  or  $-$  if  $2 \mid d'$  with  $d' \neq 0$ ,  $\epsilon = +$  otherwise. We say  $M \in \mathbb{Z}_{\text{sym}}^{n,n}$  is of  $\sigma$ -type if

$$M \equiv I_s \perp \langle 0 \rangle^{n-s} (\mathcal{N}_s)$$

for all  $s$ ,  $0 \leq s \leq n$ , and

$$M \equiv \begin{cases} I_d \perp 2I_{d'} \perp \langle 0 \rangle^{n-d-d'} (4) & \text{if } \epsilon = +, \\ I_d \perp 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{d'/2} \perp \langle 0 \rangle^{n-d-d'} (4) & \text{if } \epsilon = -. \end{cases}$$

**Proposition 3.4.** *Suppose  $\mathcal{N} \in \mathbb{Z}_+$  is odd and square-free. For each admissible type  $\sigma$  for level  $4\mathcal{N}$ , fix  $M_\sigma$  of  $\sigma$ -type. Then then with  $\sigma$  varying over all admissible types,*

$$\left\{ \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \right\}_\sigma$$

*is a set of representatives for  $\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(4\mathcal{N})$ .*

*Proof.* Fix  $\gamma = \begin{pmatrix} * & * \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$ . By Proposition 35 [7] we know there is some  $M' \in \mathbb{Z}_{\text{sym}}^{n,n}$  so that for all primes  $q \mid 4\mathcal{N}$ ,

$$M' \equiv I_{r(q)} \perp \langle 0 \rangle^{n-r(q)} (q)$$

where  $r(q) = \text{rank}_q M$ . Let  $(\mathcal{N}_0, \dots, \mathcal{N}_n)$  be the multiplicative partition of  $\mathcal{N}$  so that for each prime  $q \mid \mathcal{N}$ ,  $q \mid \mathcal{N}_s$  if and only if  $r(q) = s$ .

Set  $d = r(2)$ . By §63 [4], we know there is some  $E' \in SL_n(\mathbb{Z}_2)$  so that

$$E' M'^t E' = J_0 \perp 2J_1 \perp 4J$$

where  $J_0, J_1$  are unimodular over  $\mathbb{Z}_2$ ,  $J_0$  is  $d \times d$ , and

$$J_1 = I_{d''} \perp \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \perp \dots \perp \begin{pmatrix} a_r & b_r \\ b_r & c_r \end{pmatrix}$$

with  $a_i c_i - b_i^2 \not\equiv 0 \pmod{2}$  ( $1 \leq i \leq r$ ). Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \equiv I \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (2).$$

So adjusting  $E'$ , we can assume that  $J_1 \equiv I_\ell \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m (2)$  for some  $\ell, m \in \mathbb{Z}_{\geq 0}$ . We also have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \equiv I (2).$$

Hence, further adjusting  $E'$ , we can assume that

$$J_1 \equiv I_{d'} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{d'/2} (2)$$

where  $J_1$  is  $d' \times d'$ .

Now (using Lemma 6.1 [7]) choose  $E \in SL_n(\mathbb{Z})$  so that  $E \equiv I \pmod{\mathcal{N}}$  and  $E \equiv E' \pmod{4}$ . Thus

$$\begin{pmatrix} I & 0 \\ M'' & I \end{pmatrix} = \begin{pmatrix} {}^t E^{-1} & \\ & E \end{pmatrix} \begin{pmatrix} I & 0 \\ M' & I \end{pmatrix} \begin{pmatrix} {}^t E & \\ & E^{-1} \end{pmatrix} \in \Gamma_\infty \gamma \Gamma_0(4\mathcal{N})$$

with  $M'' \equiv M' \pmod{\mathcal{N}}$ ,  $M'' \equiv J_0 \perp 2J_1 \perp \langle 0 \rangle^{n-d-d'} \pmod{4}$  where  $J_0, J_1$  are symmetric and invertible modulo 2, and  $J_1 \equiv I_{d'}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{d'/2} \pmod{2}$ . Now take  $\bar{J}_0 \in \mathbb{Z}_{\text{sym}}^{n,n}$  so that  $J_0 \bar{J}_0 \equiv I \pmod{4}$ . Take  $\delta \in \Gamma_0(4\mathcal{N})$  so that  $\delta \equiv I \pmod{\mathcal{N}}$  and

$$\delta \equiv \begin{pmatrix} \bar{J}_0 & & \bar{J}_0 - I & \\ & I_{n-d} & & 0 \\ & & J_0 & \\ & & & I_{n-d} \end{pmatrix} \pmod{4}.$$

Set  $\epsilon = +$  if  $J_1 \equiv I_{d'} \pmod{2}$ , and set  $\epsilon = -$  otherwise. Then

$$\begin{pmatrix} I & 0 \\ M'' & I \end{pmatrix} \delta \equiv \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \pmod{4\mathcal{N}}$$

where  $\sigma = ((\mathcal{N}_0, \dots, \mathcal{N}_n), (d, d', \epsilon))$ . Hence by Proposition 3.3 [7],

$$\gamma \in \Gamma_\infty \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \Gamma_0(4\mathcal{N}).$$

Thus  $\cup_\sigma \Gamma_\infty \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} = Sp_n(\mathbb{Z})$ .

Now we want to show the above union is disjoint. So suppose that  $M_\sigma$  and  $M_{\sigma'}$  are  $\sigma$ - and  $\sigma'$ -type (respectively) where  $\sigma((\mathcal{N}_0, \dots, \mathcal{N}_n), (d, d', \epsilon))$  and  $\sigma'((\mathcal{N}'_0, \dots, \mathcal{N}'_n), (r, r', \epsilon'))$ , and suppose that

$$\begin{pmatrix} I & 0 \\ M_{\sigma'} & I \end{pmatrix} \in \Gamma_\infty \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \Gamma_0(4\mathcal{N}).$$

Thus

$$\begin{pmatrix} I & 0 \\ M_{\sigma'} & I \end{pmatrix} = \begin{pmatrix} {}^t E^{-1} & * \\ 0 & E \end{pmatrix} \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some  $E \in GL_n(\mathbb{Z})$  and  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4\mathcal{N})$ . So for all primes  $q|4\mathcal{N}$ ,  $\text{rank}_q M_{\sigma'} = \text{rank}_q M_\sigma$ . This means that  $\mathcal{N}'_i = \mathcal{N}_i$  for  $0 \leq i \leq n$  and  $r = d$ . Write

$$E = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where  $E_1, A_1$  are  $d \times d$ . We have

$$M_\sigma \equiv I_d \perp 2J \pmod{4} \text{ and } M_{\sigma'} \equiv I_d \perp 2J' \pmod{4}$$

where  $J = J_1 \perp \langle 0 \rangle^{n-d-d'}$ ,  $J' = J'_1 \perp \langle 0 \rangle^{n-d-r'}$ ,  $J_1$  is  $d' \times d'$ ,  $J'_1$  is  $r' \times r'$ , and  $J_1, J'_1$  are invertible modulo 2. We have  $M_{\sigma'} \equiv EM_\sigma A \pmod{4}$ , so  $E_1, A_1$  are invertible modulo 2,  $E_3, A_2 \equiv 0 \pmod{2}$ ,  $E_4, A_4$  are invertible modulo 2, and  $J' \equiv E_4 J A_4 \pmod{2}$ . Hence  $\text{rank}_2 J' = \text{rank}_2 J$ , meaning that  $r' = d'$ . Writing  $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$  where  $D_1$  is  $d \times d$ , and knowing that  $E(M_\sigma B + D) = I$ , we see that we must have  $D_3 \equiv 0 \pmod{2}$  and  $E_4 D_4 \equiv I \pmod{2}$ . We also know

that  $A^t D \equiv I \pmod{4\mathcal{N}}$ , so  $A_4^t D_4 \equiv I \pmod{2}$ . Thus  $E_4 \equiv {}^t A_4 \pmod{2}$ . Hence  $J' \equiv {}^t A_4 J A_4 \pmod{2}$ . Now suppose that  $d' \geq 1$  and write  $A_4 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$  where  $\alpha_1$  is  $d' \times d'$ . So

$$\begin{pmatrix} J'_1 & \\ & 0 \end{pmatrix} \equiv \begin{pmatrix} {}^t \alpha_1 J_1 \alpha_1 & {}^t \alpha_1 J_1 \alpha_2 \\ {}^t \alpha_2 J_1 \alpha_1 & {}^t \alpha_2 J_1 \alpha_2 \end{pmatrix} \pmod{2},$$

and hence  $\alpha_1$  is invertible modulo 2,  $\alpha_2 \equiv 0 \pmod{2}$ . Thus over  $\mathbb{Z}/2\mathbb{Z}$ ,  $J_1$  and  $J'_1$  represent the same quadratic form. Noting that  $I_{d'}$  has an anisotropic vector modulo 2 and for any  $\ell \geq 1$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\ell$  does not, we must have  $\epsilon' = \epsilon$ .

Thus  $\sigma' = \sigma$  whenever  $\begin{pmatrix} I & 0 \\ M_{\sigma'} & I \end{pmatrix} \in \Gamma_\infty \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix} \Gamma_0(4\mathcal{N})$ .  $\square$

Next, for  $\gamma \in \Gamma_0(4)$ , we determine necessary conditions on  $\chi$  to have  $\mathbb{E}_\gamma \neq 0$ . Then for certain  $\gamma \notin \Gamma_0(4)$ , we show that  $\mathbb{E}_\gamma = 0$  regardless of choices for  $\chi$  and  $\varphi_\gamma(\tau)$ .

**Proposition 3.5.** *Suppose that  $\mathcal{N} \in \mathbb{Z}_+$  is odd and square-free, and suppose that  $\sigma = ((\mathcal{N}_0, \dots, \mathcal{N}_n), (0, 0, +))$  is an admissible type for level  $4\mathcal{N}$ . Take  $M_\sigma$  of  $\sigma$ -type, and set  $\gamma = \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix}$ . Thus  $\gamma \in \Gamma_0(4)$  and  $\mathbb{E}_\gamma = 0$  unless  $\chi_q^2 = 1$  for all primes  $q$  dividing  $\mathcal{N}_1 \cdots \mathcal{N}_{n-1}$ . (Note that we necessarily have  $\chi_4^2 = 1$ .)*

*Proof.* Take  $q$  to be a prime dividing  $\mathcal{N}_1 \cdots \mathcal{N}_{n-1}$ . Hence  $M_\sigma \equiv \begin{pmatrix} I_d & \\ & 0 \end{pmatrix} \pmod{q}$  where  $0 < d < n$ . Let  $u \in \mathbb{Z}$  be a unit modulo  $q$ , with  $\bar{u} \in \mathbb{Z}$  so that  $u\bar{u} \equiv 1 \pmod{q}$ . By Lemma 6.1 [7], we can take  $E \in SL_n(\mathbb{Z})$  and  $\delta \in Sp_n(\mathbb{Z})$  so that  $E \equiv I \pmod{q}$ ,  $E \equiv \begin{pmatrix} u & & \\ & I & \\ & & \bar{u} \end{pmatrix} \pmod{q}$ ,  $\delta \equiv I \pmod{q}$ ,

$$\delta \equiv \begin{pmatrix} \bar{u} & & w & & \\ & I_{n-2} & & 0 & \\ & & \bar{u} & & 0 \\ & & & u & \\ & & & & I_{n-2} \\ & & & & & u \end{pmatrix} \pmod{q}$$

where  $w = \bar{u} - u$ . Thus  $\delta \in \Gamma_0(4\mathcal{N})$ ,  $\beta = \begin{pmatrix} {}^t E^{-1} & \\ & E \end{pmatrix} \in \Gamma_\infty$ , and  $\beta\gamma\delta\gamma^{-1} \in \Gamma_\infty\Gamma(4\mathcal{N})$ . Hence  $\delta \in \Gamma_\gamma$ . Since  $\gamma \in \Gamma_0(4)$ , we have that  $\zeta_\gamma(\delta) = 1$ . So by Lemma 3.1, if  $\mathbb{E}_\gamma \neq 0$  then  $1 = \chi(\delta) = \chi_q^2(u)$ . This argument holds for all units modulo  $q$ , and for all primes  $q$  dividing  $\mathcal{N}_1 \cdots \mathcal{N}_{n-1}$ , proving the proposition.  $\square$

**Proposition 3.6.** *Suppose  $\mathcal{N} \in \mathbb{Z}_+$  is odd and square-free, and  $\sigma = ((\mathcal{N}_0, \dots, \mathcal{N}_n), (d, d', +))$  is an admissible type for level  $4\mathcal{N}$  with  $d' > 0$ . Then with  $\chi$  a character modulo  $4\mathcal{N}$ ,  $M_\sigma$  of  $\sigma$ -type, and  $\gamma = \begin{pmatrix} 1 & 0 \\ M_\sigma & 1 \end{pmatrix}$ , we have  $\mathbb{E}_\gamma = 0$ , regardless of the choice of automorphy factor for  $\gamma$ .*

*Proof.* Fix a choice of character  $\chi$  modulo  $4\mathcal{N}$  and an automorphy factor  $\varphi_\gamma(\tau)$  for  $\gamma$ . We have  $M_\sigma \equiv \langle m_1, \dots, m_n \rangle \pmod{4\mathcal{N}}$  where

$$m_{d+1} \equiv \begin{cases} 0 \pmod{\mathcal{N}_0 \cdots \mathcal{N}_d}, \\ 1 \pmod{\mathcal{N}_{d+1} \cdots \mathcal{N}_n}, \\ 2 \pmod{4}. \end{cases}$$

So (using Proposition 3.3) we can assume that  $M_\sigma = \langle m_1, \dots, m_n \rangle$ .

Now set  $m = m_{d+1}$ ,

$$A = \begin{pmatrix} I_d & & \\ & 1 + 2\mathcal{N} & \\ & & I_{n-d-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0_d & & \\ & 2\mathcal{N}/m & \\ & & 0_{n-d-1} \end{pmatrix},$$

$$C = \begin{pmatrix} 0_d & & \\ & -2\mathcal{N}m & \\ & & 0_{n-d-1} \end{pmatrix}, \quad D = \begin{pmatrix} I_d & & \\ & 1 - 2\mathcal{N} & \\ & & I_{n-d-1} \end{pmatrix}.$$

One easily checks that with

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we have  $(M_\sigma I)\alpha = (M_\sigma I)$  and consequently  $\alpha \in \Gamma_\gamma$ . As we saw in our construction of Eisenstein series, we have  $\mathbb{E}_\gamma = 0$  unless  $\Gamma'_{\gamma, \chi} = \Gamma_\gamma$ . Then as in the proof of Lemma 3.1, we have

$$(\zeta_\gamma(\alpha))^2 = \frac{(\overline{\mathcal{G}}_{-2\mathcal{N}m}(1 - 2\mathcal{N}))^2}{1 - 2\mathcal{N}} = \frac{(\overline{\mathcal{G}}_{2\mathcal{N}m}(2\mathcal{N} - 1))^2}{1 - 2\mathcal{N}}.$$

Since  $\mathcal{N}$  is odd, we have  $2\mathcal{N} - 1 \equiv 1 \pmod{4}$  and hence  $(\overline{\mathcal{G}}_{2\mathcal{N}m}(2\mathcal{N} - 1))^2 = 2\mathcal{N} - 1$ . Therefore  $(\zeta_\gamma(\alpha))^2 = -1$  and so  $\zeta_\gamma(\alpha) = \pm i$ . We also have  $\chi(\alpha) = \chi(1 - 2\mathcal{N}) = \chi_4(-1) = \pm 1$ . Consequently,  $\chi \zeta_\gamma^k(\alpha) = \pm i$ . Hence  $\Gamma'_{\gamma, \chi} \neq \Gamma_\gamma$ , and so  $\mathbb{E}_\gamma = 0$ .  $\square$

Using elements of the above proof, we prove the following.

**Corollary 3.7.** *Let  $\tilde{\Gamma}_0(4) = \{\tilde{\gamma} = [\gamma, \theta(\gamma\tau)/\theta(\tau)] : \gamma \in \Gamma_0(4)\}$ . Suppose that  $\Gamma$  is a subgroup of  $Sp_n(\mathbb{Z})$  containing  $\Gamma_0(4)$ , and that  $\tilde{\Gamma}$  is a cover of  $\Gamma$ , meaning that  $\tilde{\Gamma}$  is a group whose elements are of the form  $[\gamma, \varphi_\gamma(\tau)]$ ,  $\gamma \in \Gamma$  and  $\varphi_\gamma(\tau)$  an automorphy factor for  $\gamma$ . Suppose further that  $\tilde{\Gamma}_0(4) \subseteq \tilde{\Gamma}$  and that  $\begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \in \Gamma$  with  $M \equiv I_d \perp 2I_{d'} \perp \langle 0 \rangle^{n-d-d'} \pmod{4}$  where  $d + d' > 0$ , or  $M \equiv I_d \perp \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}^{d'/2} \perp \langle 0 \rangle^{n-d-d'} \pmod{4}$  where  $d > 0$ . Then  $\tilde{\Gamma}$  is not a one-fold cover of  $\Gamma$ . In particular, when  $n = 1$ , there is no group  $\Gamma$  so that  $\Gamma_0(4) \subsetneq \Gamma \subseteq SL_2(\mathbb{Z})$  and  $\tilde{\Gamma}$  is a one-fold cover of  $\Gamma$ .*

*Proof.* With  $M$  as above, we have

$$\begin{pmatrix} I & 0 \\ M & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ M & I \end{pmatrix}^2 \in \Gamma;$$



consequently  $\Gamma$  contains a matrix  $\gamma = \begin{pmatrix} I & 0 \\ M' & I \end{pmatrix}$  where  $M' \equiv 2I_r \perp \langle 0 \rangle^{n-r}$  (4) for some  $r > 0$ . Take an automorphy factor  $\varphi_\gamma(\tau)$  for  $\gamma$  so that  $[\gamma, \varphi_\gamma(\tau)] \in \tilde{\Gamma}$ . Then since  $\tilde{\Gamma}$  is a group with identity  $[I, 1]$ , we have

$$\tilde{\gamma}^{-1} = [\gamma^{-1}, 1/\varphi_\gamma(\gamma^{-1}\tau)] \in \tilde{\Gamma}.$$

Then from the proof of Proposition 3.7, there is some  $\alpha \in \Gamma_\gamma \subseteq \Gamma_0(4)$  so that

$$\tilde{\gamma}\tilde{\alpha}\tilde{\gamma}^{-1} = [I, \zeta_\gamma(\alpha)](\widetilde{\gamma\alpha\gamma^{-1}}) \neq \widetilde{\gamma\alpha\gamma^{-1}}.$$

Thus there are (at least) two distinct elements in  $\tilde{\Gamma}$  of the form  $[\gamma\alpha\gamma^{-1}, *]$ . Hence  $\tilde{\Gamma}$  is not a one-fold cover of  $\Gamma$ .  $\square$

**Remark.** It would be interesting to know whether there is a group  $\Gamma$  with  $\Gamma_0(4) \subsetneq \Gamma \subseteq Sp_n(\mathbb{Z})$  so that  $\tilde{\Gamma}$  is a one-fold cover of  $\Gamma$ . Note that by the above result and Proposition 3.5, such a group  $\Gamma$  would necessarily have an element of the form

$$\begin{pmatrix} I & 0 \\ M' & I \end{pmatrix}$$

where  $M' \equiv \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}^r \perp \langle 0 \rangle^{n-r/2}$  (4) where  $r > 0$ .

#### 4. THE ACTION OF HECKE OPERATORS ATTACHED TO ODD PRIMES WHERE THE LEVEL IS $4\mathcal{N}$ WITH $\mathcal{N}$ ODD AND SQUARE-FREE

Throughout, we fix  $n, k, \mathcal{N} \in \mathbb{Z}_+$  with  $k$  odd,  $n > (k+1)/2$ ,  $\mathcal{N}$  odd and square-free, and  $\chi$  a character modulo  $4\mathcal{N}$ .

Here we look at the action on Siegel Eisenstein series of Hecke operators. However, because of the constraints reflected in Proposition 2.2, we are only able to do this satisfactorily for Hecke operators attached to odd primes, and for Eisenstein series supported on the  $\Gamma_0(4\mathcal{N})$ -orbit of a matrix  $\gamma \in \Gamma_0(4)$ . (In the process of our evaluation, we point out where we find these restrictions necessary.) We restrict our attention to level  $4\mathcal{N}$  with  $\mathcal{N}$  odd and square-free so that we can evaluate the action of the Hecke operators attached to primes dividing  $\mathcal{N}$ .

In Proposition 3.4 we presented a set of representatives for the double quotient  $\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(4\mathcal{N})$ . From that we can see that a representatives for  $\Gamma_\infty \backslash \Gamma_0(4) / \Gamma_0(4\mathcal{N})$  are associated to admissible types  $((\mathcal{N}_0, \dots, \mathcal{N}_n), (0, 0, +))$  where  $(\mathcal{N}_0, \dots, \mathcal{N}_n)$  is a (multiplicative) partition of  $\mathcal{N}$ . Hence to ease our notation, for  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$  a partition of  $\mathcal{N}$ , we fix a diagonal matrix  $M_\sigma \in 4\mathbb{Z}^{n,n}$  with  $M_\sigma \equiv I_s \perp \langle 0 \rangle^{n-s} (\mathcal{N}_s)$  for all  $s$ ,  $0 \leq s \leq n$ . Then we set  $\gamma_\sigma = \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix}$ , and we often write  $\mathbb{E}_\sigma$  for  $\mathbb{E}_{\gamma_\sigma}$ . Note that with such  $\sigma$  and  $M_\sigma$ , we have that

$$GL_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4\mathcal{N}) = SL_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4\mathcal{N}),$$

so we can write  $\mathbb{E}_\sigma$  as a sum over representatives

$$SL_n(\mathbb{Z})(M \ N) \in SL_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4\mathcal{N}).$$

This observation is useful in allowing us to more easily adapt results from [7], as with the following result.

**Proposition 4.1.** *Fix a multiplicative partition  $\sigma$  of  $\mathcal{N}$ , and suppose  $\mathbb{E}_{\gamma_\sigma} \neq 0$ . Take  $(M \ N) \in SL_n(\mathbb{Z})(M_\sigma \ I)\gamma$  where  $\gamma \in \Gamma_0(\mathcal{N})$ , and fix a prime  $q|\mathcal{N}$ . There are  $E_0, E_1 \in SL_n(\mathbb{Z})$  so that*

$$E_0 M E_1 \equiv \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \pmod{q}$$

with  $M_1$  invertible modulo  $q$ ; for any such  $E_0, E_1$  we have

$$E_0 N {}^t E_1^{-1} \equiv \begin{pmatrix} N_1 & N_2 \\ 0 & N_4 \end{pmatrix} \pmod{q}$$

and  $\chi_q(\gamma) = \chi_q(\det \overline{M}_1 \cdot \det N_4)$ . Also,  $\chi_4(M, N) = \chi_4(\det N)$ . Further, for any  $G \in GL_n(\mathbb{Z})$ , we have

$$\chi(GM, GN) = \chi(\det G)\chi(M, N) = \chi(MG, N {}^t G^{-1}).$$

*Proof.* Essentially, this is Proposition 3.7 from [7], although for that proposition the level is  $\mathcal{N}$  with  $\mathcal{N}$  square-free. So here the difference is that our level is  $4\mathcal{N}$  with  $\mathcal{N}$  odd and square-free. Hence for  $\gamma \in \Gamma_0(4\mathcal{N})$  so that  $(M \ N) \in GL_n(\mathbb{Z})(M_\sigma \ I)\gamma$ , we have  $\chi_4(\gamma) = \chi_4(\det N)$ .  $\square$

We begin by evaluating  $\mathbb{E}_\sigma |T_j(q^2)$  ( $1 \leq j \leq n$ ) for  $\sigma$  a partition of  $\mathcal{N}$  and  $q$  a prime dividing  $\mathcal{N}$ . We show that the span of these  $\mathbb{E}_\sigma$  has a basis

$$\{\tilde{\mathbb{E}}_\sigma : \sigma \text{ a partition of } \mathcal{N} \}$$

so that each  $\tilde{\mathbb{E}}_\sigma$  is a simultaneous eigenform for  $\{T_n(q^2) : \text{prime } q|\mathcal{N}\}$ . Further, we show that for  $\sigma, \rho$  distinct partitions of  $\mathcal{N}$ , there is some prime  $q|\mathcal{N}$  so that the  $T_n(q^2)$ -eigenvalues of  $\tilde{\mathbb{E}}_\sigma$  and  $\tilde{\mathbb{E}}_\rho$  differ. Then, since the Hecke operators commute, we can show that each  $\tilde{\mathbb{E}}_\sigma$  is an eigenform for all Hecke operators associated to odd primes, and we explicitly compute the eigenvalues of all  $T_j(p^2)$  where  $1 \leq j \leq n$  and  $p$  an odd prime.

From Proposition 1.3 and Theorem 2.3 of [6], and Proposition 2.1 of [7], we have the following. Note that here we have normalized the Hecke operator presented in [6]; also, we have simplified a Gauss sum that appeared there.

**Proposition 4.2.** *Suppose  $F$  is a modular form of degree  $n$ , weight  $k/2$ , level  $4\mathcal{N}$  and character  $\chi$ .*

(a) *For  $q$  a prime dividing  $4\mathcal{N}$  and  $1 \leq j \leq n$ , we have*

$$F|T_j(q^2) = q^{j(k/2-n-1)} \sum_{G,Y} F \left[ \begin{pmatrix} X_j^{-1}G^{-1} & X_j^{-1}Y {}^t G \\ X_j {}^t G & \end{pmatrix}, q^{j/2} \right]$$

where  $G$  varies over  $\mathcal{K}_j(q)$ , and  $Y$  varies over  $\mathcal{Y}_j(q^2)$ , meaning that  $Y = \begin{pmatrix} U & V \\ {}^t V & 0 \end{pmatrix}$  so that  $U$  varies over integral symmetric  $j \times j$  matrices modulo  $q^2$ , and  $V$  varies over integral  $j \times (n-j)$  matrices modulo  $q$ .

(b) For  $p$  a prime not dividing  $4N$  and  $1 \leq j \leq n$ , we have

$$\begin{aligned} & F|T_j(p^2) \\ &= p^{j(k/2-n-1)} \sum_{\substack{n_0, n_2 \\ G, Y}} F|\chi(p^{j-n_0+n_2}) \\ & \cdot \left[ \begin{pmatrix} X_{n_0, n_2}^{-1} G^{-1} & X_{n_0, n_2}^{-1} Y^t G \\ 0 & X_{n_0, n_2}^t G \end{pmatrix}, p^{-j/2+n_0} \mathcal{G}_{Y_1}(pI_{j-n_0-n_2}) \right]. \end{aligned}$$

Here,  $n_0, n_2 \in \mathbb{Z}_{\geq 0}$  vary so that  $n_0 + n_2 \leq j$ . For each pair  $n_0, n_2$ , we have  $G = G_1 G_2$ , where  $G_1$  varies over  $SL_n(\mathbb{Z})/\mathcal{K}_{n_0, n_2}(p)$ ,

$$G_2 = \begin{pmatrix} I_{n_0} & & \\ & G' & \\ & & I_{n_2} \end{pmatrix}$$

with  $G'$  varying over  $SL_{n'}(\mathbb{Z})/{}^t\mathcal{K}'_{j'}(p)$  where  $n' = n - n_0 - n_2$ ,  $j' = j - n_0 - n_2$ ,

$$\mathcal{K}'_{j'} = \begin{pmatrix} pI_{j'} & \\ & I \end{pmatrix} SL_{n'}(\mathbb{Z}) \begin{pmatrix} \frac{1}{p}I_{j'} & \\ & I \end{pmatrix} \cap SL_{n'}(\mathbb{Z}),$$

and  $Y$  varies over  $\mathcal{Y}_{n_0, n_2}(p^2)$ , the set of all integral, symmetric  $n \times n$  matrices

$$\begin{pmatrix} Y_0 & Y_2 & Y_3 & 0 \\ {}^tY_2 & Y_1/p & 0 & \\ {}^tY_3 & 0 & & \\ 0 & & & \end{pmatrix}$$

with  $Y_0$   $n_0 \times n_0$ , varying modulo  $p^2$ ,  $Y_1$   $j' \times j'$ , varying modulo  $p$  provided  $p \nmid \det Y_1$ , and  $Y_2, Y_3$  varying modulo  $p$  with  $Y_3$   $n_0 \times n_2$ . Note that we can assume  $G \equiv I \pmod{4N}$  and  $Y \equiv 0 \pmod{4N}$ . Also,

$$\mathcal{G}_{Y_1}(pI_{j-n_0-n_2}) = \left( \frac{\det Y_1}{p} \right) \mathcal{G}_1(p)^{j-n_0-n_2}.$$

**Theorem 4.3.** Suppose that  $N \in \mathbb{Z}_+$  is odd and square-free,  $\chi$  is a character modulo  $4N$  so that  $\chi(-1) = 1$ . Fix a prime  $q|N$  and a multiplicative partition  $\sigma' = (N'_0, \dots, N'_n)$  of  $N/q$ . For  $0 \leq d \leq n$ , let  $\sigma_d = (N_0, \dots, N_n)$  where

$$N_i = \begin{cases} N'_i & \text{if } i \neq d, \\ qN'_d & \text{if } i = d. \end{cases}$$

Then when  $\mathbb{E}_{\sigma_d} \neq 0$  for  $0 \leq j \leq n$ , we have

$$\mathbb{E}_{\sigma_d}|T_j(q^2) = \sum_{t=0}^{n-d} A_j(d, t) \mathbb{E}_{\sigma_{d+t}}$$

where

$$\begin{aligned}
A_j(d, t) &= q^{(j-t)d-t(t+1)/2} \beta_q(d+t, t) \\
&\cdot \sum_{s=0}^j \sum_{d_5=0}^{j-s} \sum_{d_8=0}^{d_5} q^{a_j(d; s, d_5, d_8)} \overline{\chi}_{\mathcal{N}/q}(X_{s,r}^{-1} M_{\sigma_d} X_j, X_{s,r}^{-1} X_j^{-1}) \\
&\cdot \beta_q(d, s) \beta_q(t, d_5) \beta_q(n-d-t, j-s+d_8-t) \\
&\cdot \beta_q(t-d_5, d_8) \text{sym}_q^X(t-d_5-d_8) \text{sym}_q^{X'}(d_5, d_8) \\
&\cdot q^{-k(d_5-d_8)/2} \mathcal{G}_1(q)^{k(d_5-d_8)}, \\
r &= j-s-d_5+d_8, \\
a_j(d; s, d_5, d_8) &= (k/2-d)(2s+d_5-d_8) + s(s-d_8-j-1) \\
&\quad + d_8(j-d_5) - d_5(d_5+1)/2 + d_8(d_8+1)/2,
\end{aligned}$$

and  $\chi'_q(*) = \chi_q(*) \left(\frac{*}{q}\right)$ . (Here  $\text{sym}_q^X(b, c)$  is as defined in section 2.) Thus  $\widetilde{\mathbb{E}}_{\sigma_d}|T_j(q^2) = A_j(d, 0)\widetilde{\mathbb{E}}_{\sigma_d}$ .

*Proof.* Here we make use of results derived in the proof of Theorem 4.4 [7], where we evaluated the action of  $T_j(q^2)$  on integral weight Eisenstein series. The difference is that here we need to compare automorphy factors Gauss sums.

To ease notation, temporarily write  $\mathbb{E}_{d'}$  for  $\mathbb{E}_{\sigma_{d'}}$ ,  $M_{d'}$  for  $M_{\sigma_{d'}}$ ,  $X_{s,r}$  for  $X_{s,r}(q)$ ,  $\mathcal{K}_{s,r}$  for  $\mathcal{K}_{s,r}(q)$ , and  $\mathcal{Y}_j$  for  $\mathcal{Y}_j(q^2)$ .

By Proposition 4.2, we have

$$\begin{aligned}
\mathbb{E}_d(\tau)|T_j(q^2) &= q^{-j(n+1)} \sum_{(MN), G, Y} \overline{\chi}(M, N) \left( \frac{\overline{\mathcal{G}}_M(N)}{\sqrt{\det N}} \right)^{-k} \\
&\cdot S_{M,N}(X_j^{-1} G^{-1}(\tau + GY {}^t G) {}^t G^{-1} X_j^{-1})^{-k}
\end{aligned}$$

where  $SL_n(\mathbb{Z})(M \ N)$  varies over the  $\Gamma_0(4\mathcal{N})$ -orbit of  $SL_n(\mathbb{Z})(M_d \ I)$ , and  $G, Y$  vary as in Proposition 4.2. By Proposition 2.2 we know that for  $E \in SL_n(\mathbb{Z})$ , we have  $\chi(EM, EN) = \chi(M, N)$ ,  $\mathcal{G}_{EM}(EN) = \mathcal{G}_M(N)$ ,  $\sqrt{EN} = \sqrt{N}$ , and  $S_{EM, EN}(\tau) = S_{M, N}(\tau)$  for all  $\tau \in \mathfrak{h}_{(n)}$ . Thus for each  $G, Y$ , we follow [7] to adjust the pair  $(M \ N)$  by left multiplication from  $SL_n(\mathbb{Z})$  so that, with appropriate choices of  $s, r$ , we have that

$$(X_{s,r} M X_j^{-1} G^{-1}, X_{s,r} N X_j {}^t G) = 1.$$

We set  $M' = X_{s,r} M X_j^{-1} G^{-1}$ ,  $N'' = X_{s,r} N X_j {}^t G$ , and  $N' = M' GY {}^t G + N''$ . Then

$$\frac{S_{M,N}(X_j^{-1} G^{-1}(\tau + GY {}^t G) {}^t G^{-1} X_j^{-1})}{S_{M', N''}(\tau + GY {}^t G)}$$

is an analytic function whose square is  $q^{r-s-j}$  and whose limit as  $\tau \mapsto 0$  is  $q^{(r-s-j)/2}$ ; thus the quotient above is equal to  $q^{(r-s-j)/2}$ . Further, since  $q \neq 2$ , from Proposition 2.2 we know that

$$\frac{\overline{\mathcal{G}}_{M'}(N'')}{\sqrt{\det N''}} S_{M', N''}(\tau + GY {}^t G) = \frac{\overline{\mathcal{G}}_{M'}(N')}{\sqrt{\det N'}} S_{M', N'}(\tau).$$

Hence

$$\begin{aligned} \mathbb{E}_d(\tau)|T_j(q^2) &= q^{j(k/2-n-1)} \sum_{(M \ N), G, Y} \bar{\chi}(M, N) \left( \frac{\bar{\mathcal{G}}_{M'}(N')}{\sqrt{\det N'}} \right)^{-k} \\ &\quad \cdot q^{k(s-r)/2} \left( \frac{\bar{\mathcal{G}}_M(N) \cdot \sqrt{\det N''}}{\sqrt{\det N} \cdot \bar{\mathcal{G}}_{M'}(N'')} \right)^{-k} S_{M', N'}(\tau)^{-k} \end{aligned}$$

where  $SL_n(\mathbb{Z})(M \ N)$  varies over  $SL_n(\mathbb{Z})(M_d \ I)\Gamma_0(4\mathcal{N})$ ,  $G, Y$  vary as in Proposition 4.2, and  $r, s, M', N'', N'$  are as defined above. Note that we necessarily have  $\text{rank}_q M' \geq \text{rank}_q M = d$ .

Given a coprime symmetric pair  $(M' \ N')$ , we want to count how often

$$SL_n(\mathbb{Z})(M' \ N') = SL_n(\mathbb{Z})X_{s,r}(MX_j^{-1}G^{-1} \ MX_j^{-1}Y^tG + NX_j^tG)$$

for some  $(M \ N) \in SL_n(\mathbb{Z})(M_d \ I)\Gamma_0(4\mathcal{N})$ . Equivalently, we want to count how often

$$\begin{aligned} (*) \quad & X_{s,r}^{-1}E(M'GX_j \ -M'GYX_j^{-1} + N'^tG^{-1}X_j^{-1}) \\ & \in SL_n(\mathbb{Z})(M_d \ I)\Gamma_0(4\mathcal{N}) \end{aligned}$$

for some  $E \in SL_n(\mathbb{Z})$ . Since  $X_{s,r}^{-1}EX_{s,r} \in SL_n(\mathbb{Z})$  for  $E \in SL_n(\mathbb{Z})$  if and only if  $E \in \mathcal{K}_{s,r}$ , we only need to consider  $E \in \mathcal{K}_{s,r} \setminus SL_n(\mathbb{Z})$ . Thus

$$\mathbb{E}_d(\tau)|T_j(q^2) = \sum_{(M' \ N')} c_d(M', N') \bar{\chi}(M', N') \left( \frac{\bar{\mathcal{G}}_{M'}(N')}{\sqrt{\det N'}} S_{M', N'}(\tau) \right)^{-k}$$

where

$$\begin{aligned} c_d(M', N') &= q^{j(k/2-n-1)} \chi(M', N') \sum_{s,r,E,G,Y} \bar{\chi}(M, N) q^{k(s-r)/2} \\ &\quad \cdot \left( \frac{\bar{\mathcal{G}}_M(N) \sqrt{\det X_{s,r}NX_j}}{\sqrt{\det N} \bar{\mathcal{G}}_{X_{s,r}MX_j^{-1}(X_{s,r}NX_j)}} \right)^{-k}, \end{aligned}$$

$s, r \in \mathbb{Z}_{\geq 0}$ ,  $E \in \mathcal{K}_{s,r} \setminus SL_n(\mathbb{Z})$ ,  $G \in SL_n(\mathbb{Z})/\mathcal{K}_j$ ,  $Y \in \mathcal{Y}_j$  such that

$$\begin{aligned} (M \ N) &= X_{s,r}^{-1}E(M'GX_j \ M'GYX_j^{-1} + N'^tG^{-1}X_j^{-1}) \\ &\in SL_n(\mathbb{Z})(M_d \ I)\Gamma_0(4\mathcal{N}). \end{aligned}$$

(Here we have used that for  $G \in SL_n(\mathbb{Z})$  we have  $\mathcal{G}_{M'}(N') = \mathcal{G}_{M'G}(N'^tG^{-1})$ .) We also know that  $\mathbb{E}_d|T_j(q^2)$  is a modular form, and hence is a linear combination of  $\mathbb{E}_{d'}$  for  $d' \geq d$ . Thus  $\mathbb{E}_d|T_j(q^2) = \sum_{d' \geq d} c_d(M_{d'}, I)\mathbb{E}_{d'}$ .

As shown in the proof of Theorem 4.4 [7], given  $s, r$ , each solution  $E, G, Y$  to (\*) corresponds to choices for  $s, d_5, d_7, d_8$  so that  $s \leq d$ ,  $d' = d + d_5 + d_7 + d_8$  and  $M, N$  have the following forms.

$$M = \begin{pmatrix} A'_1 & qA_1 & qA'_2 & qA_2 \\ qA'_3 & qA_3 & A'_4 & qA_4 \\ qA'_5 & qA_5 & qA'_6 & qA_6 \\ q^2A'_7 & q^2A_7 & qA'_8 & qA_8 \end{pmatrix}, \quad N = \begin{pmatrix} D'_1 & D_1 & D'_2 & D_2 \\ D'_3 & D_3 & D'_4 & D_4 \\ qD'_5 & D_5 & qD'_6 & D_6 \\ qD'_7 & D_7 & qD'_8 & qD_8 \end{pmatrix}$$

where  $A'_1, D'_1$  are  $s \times s$ ,  $A'_4, D'_4$  are  $(d-s) \times (d-s)$ ,  $A_7, D_7$  are  $r \times (j-s)$ ,  $A'_1, A'_4$  are invertible modulo  $q$ . Since  $(M, N) = 1$ , we must have

(row)  $\text{rank}_q \begin{pmatrix} D_5 & D_6 \\ D_7 & 0 \end{pmatrix} = n - d$ , and since  $(X_{s,r}MX_j^{-1} \ X_{s,r}NX_j) = 1$ , we must have (row)  $\text{rank}_q \begin{pmatrix} A_5 & 0 & 0 & D_6 \\ A_7 & A_8 & D_7 & 0 \end{pmatrix} = n - d$ . So  $\text{rank}_q D_7 = r$  and  $\text{rank}_q(A_5 \ D_6) = n - d - r$ . Further, adjusting  $E$  by left multiplication from  $\mathcal{K}_{s,r}$  and  $G$  by right multiplication from  $\mathcal{K}_j$ , we can assume that modulo  $q$ ,

$$A_5 \equiv \begin{pmatrix} \alpha_5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_7 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_8 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha_8 & 0 \end{pmatrix}$$

where  $\alpha_5$  is  $d_5 \times d_5$ ,  $\alpha_7$  is  $d_7 \times d_7$ ,  $\alpha_8$  is  $d_8 \times d_8$ , and  $\alpha_5, \alpha_7, \alpha_8$  are invertible modulo  $q$ . (So we necessarily have  $d_5 + d_7 \leq j - s$  and  $d_8 \leq n - j - d + s$ .) Also, as  $(M \ N)$  is a coprime symmetric pair, modulo  $q$  we have

$$D_5 \equiv \begin{pmatrix} \beta'_1 & * & * \\ 0 & * & * \end{pmatrix}, \quad D_6 \equiv \begin{pmatrix} \gamma'_1 & * \\ 0 & \gamma_4 \end{pmatrix},$$

$$D_7 \equiv \begin{pmatrix} 0 & \delta_2 & 0 \\ 0 & * & \delta'_6 \\ \delta'_7 & * & * \end{pmatrix}$$

where  $\beta'_1$  is  $d_5 \times d_5$ ,  $\gamma'_1$  is  $d_5 \times d_8$ ,  $\gamma_4$  is  $(n - d - r - d_5) \times (n - j - d + s - d_8)$ ,  $\delta_2$  is  $(r - d_7 - d_8) \times (j - s - d_5 - d_7)$ ,  $\delta'_6$  is  $d_7 \times d_7$ , and  $\delta'_7$  is  $d_8 \times d_5$ . Then a careful analysis in the proof of Theorem 4.4 [7] tells us that  $d_5 \geq d_8$ ,  $r = j - s - d_5 + d_8$ , and  $\gamma_4, \delta_2, \delta'_6, \begin{pmatrix} \beta'_1 & \gamma'_1 \\ \delta'_7 & 0 \end{pmatrix}$  are square and invertible modulo  $q$ .

From the above descriptions of  $M, N$ , we can see that

$$(X_{0,r}MX_j^{-1}X_s \ X_{0,r}NX_jX_s^{-1})$$

is an integral, coprime pair (which is necessarily symmetric). Thus by Propositions 2.2(c) and 5.1, we have

$$\mathcal{G}_{X_{s,r}MX_j^{-1}}(X_{s,r}NX_j) = q^s \mathcal{G}_{X_{0,r}MX_j^{-1}X_s}(X_{0,r}NX_jX_s^{-1}).$$

Let

$$P_1 = \begin{pmatrix} 0 & I_s & 0 \\ I_{j-s} & 0 & 0 \\ 0 & 0 & I_{n-j} \end{pmatrix}.$$

So using Proposition 2.2(c) and recalling that  $P_1^{-1} = {}^tP_1$ , we have

$$\begin{aligned} \mathcal{G}_{X_{0,r}MX_j^{-1}X_s}(X_{0,r}NX_jX_s^{-1}) &= \mathcal{G}_{X_{0,r}MX_j^{-1}X_sP_1}(X_{0,r}NX_jX_s^{-1}P_1) \\ &= \mathcal{G}_{X_{0,r}MP_1X_j^{-1}}(X_{0,r}NP_1X_{j-s}). \end{aligned}$$

To prepare to use Proposition 5.2, choose  $E_0, G_0 \in SL_{d_5}(\mathbb{Z})$  so that

$$E_0\gamma'_1 \equiv \begin{pmatrix} 0 \\ \gamma''_1 \end{pmatrix} (q), \quad \delta'_7 G_0 \equiv \begin{pmatrix} 0 & \delta''_7 \end{pmatrix} (q)$$

where  $\gamma_1'', \delta_7''$  are  $d_8 \times d_8$ . Write

$$E_0 \beta_1' G_0 = \begin{pmatrix} \beta_1'' & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$$

where  $\beta_1''$  is  $(d_5 - d_8) \times (d_5 - d_8)$ . So

$$\begin{pmatrix} E_0 & \\ & I_{d_8} \end{pmatrix} \begin{pmatrix} \beta_1' & \gamma_1' \\ \delta_7' & 0 \end{pmatrix} \begin{pmatrix} G_0 & \\ & I_{d_8} \end{pmatrix} \equiv \begin{pmatrix} \beta_1'' & \rho_2 & 0 \\ \rho_3 & \rho_4 & \gamma_1'' \\ 0 & \delta_7'' & 0 \end{pmatrix} (q),$$

and since this matrix is invertible modulo  $q$ , we must have  $\text{rank}_q \delta_7'' = d_8 = \text{rank}_q \gamma_1''$ , and  $\text{rank}_q \beta_1'' = d_5 - d_8$ . (So  $\beta_1'', \gamma_1'', \delta_7''$  are invertible modulo  $q$ .) Write

$$E_0 \alpha_5' {}^t G_0^{-1} = \begin{pmatrix} \alpha_5' & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}$$

where  $\alpha_5'$  is  $(d_5 - d_8) \times (d_5 - d_8)$ . By the symmetry of  $M {}^t N$ , we have that

$$\begin{pmatrix} E_0 & \\ & I_{d_8} \end{pmatrix} \begin{pmatrix} \beta_1' & \gamma_1' \\ \delta_7' & 0 \end{pmatrix} \begin{pmatrix} {}^t \alpha_5 & \\ & {}^t \alpha_8 \end{pmatrix} \begin{pmatrix} {}^t E_0 & \\ & I_{d_8} \end{pmatrix}$$

is symmetric modulo  $q$ ; consequently  $\omega_2 \equiv 0 (q)$  and so  $\alpha_5', \omega_4$  are invertible modulo  $q$ . Now set

$$E = \begin{pmatrix} I_d & & \\ & E_0 & \\ & & I_{n-d-d_5} \end{pmatrix}, G = \begin{pmatrix} G_0 & & \\ & I_{n-d_5} & \end{pmatrix}, P_2 = \begin{pmatrix} 0 & I_{d_5-d_8} & 0 \\ I_r & 0 & 0 \\ 0 & 0 & I_{n-j-s} \end{pmatrix}.$$

Then  $E$  commutes with  $X_{0,r}$  (since  $d_8 \leq n - j - d + s$  and hence  $r = j - s - d_5 + d_8 \leq n - d - d_5$ ). Somewhat similarly,  $G$  and  $P_2$  commute with  $X_{j-s}$  (since  $d_5 \leq j - s$  and  $r + d_5 - d_8 = j - s$ ). Thus by Proposition 2.2(c),

$$\begin{aligned} & \mathcal{G}_{X_{0,r} M P_1 X_{j-s}^{-1}} (X_{0,r} N P_1 X_{j-s}) \\ &= \mathcal{G}_{X_{0,r} E M P_1 {}^t G^{-1} P_2 X_{j-s}^{-1}} (X_{0,r} E N P_1 G P_2 X_{j-s}). \end{aligned}$$

Hence with  $\widetilde{M} = E M P_1 {}^t G^{-1}$  and  $\widetilde{N} = E N P_1 G$ , we know that

$$\begin{aligned} & (\widetilde{M} P_2 X_{j-s}^{-1} X_r \quad \widetilde{N} P_2 X_{j-s} X_r^{-1}) \text{ is an integral coprime pair} \\ \iff & (\widetilde{M} P_2 X_{j-s}^{-1} X_r {}^t P_2 \quad \widetilde{N} P_2 X_{j-s} X_r^{-1} {}^t P_2) \text{ is an integral coprime pair} \\ \iff & (\widetilde{M} X_{j-s-r}^{-1} \quad \widetilde{N} X_{j-s-r}) \text{ is an integral coprime pair.} \end{aligned}$$

Recall that  $P_1$  has permuted the 1st  $j - s$  columns of  $M$  with the next  $s$  columns of  $M$ , and similarly for  $N$ . (So, for instance, the top row of blocks of  $M P_1$  is  $(q A_1 \ A_1' \ q A_2' \ q A_2)$  and hence  $\widetilde{M} X_{j-s-r}^{-1}$  is integral.) Using our block decompositions of  $M$  and  $N$  in terms of subscripted  $A$ s and  $D$ s, we have the following. In  $\widetilde{M} X_{j-s-r}^{-1}$ , the  $(1, 2)$  block is  $A_1'$  which is  $s \times s$  and invertible modulo  $q$ , and the  $(2, 3)$  block is  $A_4'$  which is  $(d - s) \times (d - s)$  and invertible modulo  $q$ . Let  $\widetilde{A}_5$  denote the  $(3, 1)$  block of  $\widetilde{M} X_{j-s-r}^{-1}$ , and  $\widetilde{D}_5, \widetilde{D}_6$  and  $\widetilde{D}_7$  the  $(3, 1), (3, 4)$  and  $(4, 1)$  blocks of  $\widetilde{N} X_{j-s-r}$ . Then modulo

$q$ , we have

$$\begin{aligned}\tilde{A}_5 &\equiv \begin{pmatrix} \alpha'_5 & 0 & 0 & 0 \\ \omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_5 \equiv \begin{pmatrix} q\beta''_1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad \tilde{D}_6 \equiv \begin{pmatrix} 0 & * \\ \gamma''_1 & * \\ 0 & \gamma_4 \end{pmatrix}, \\ \tilde{D}_7 &\equiv \begin{pmatrix} 0 & 0 & \delta_2 & 0 \\ 0 & 0 & * & \delta'_6 \\ 0 & \delta''_7 & * & * \end{pmatrix},\end{aligned}$$

where (as we've previously noted)  $\alpha'_5, \beta''_1, \gamma''_1, \gamma_4, \delta_2, \delta'_6, \delta''_7$  are square and invertible modulo  $q$ . Hence  $(\tilde{M}X_{j-s-r}^{-1} \tilde{N}X_{j-s-r})$  has  $q$ -rank  $n$ , and thus is a coprime symmetric pair. So by Proposition 5.2 we have

$$\begin{aligned}\mathcal{G}_{X_{0,r}\tilde{M}P_2X_{j-s}^{-1}}(X_{0,r}\tilde{N}P_2X_{j-s}) &= \mathcal{G}_{\tilde{M}P_2X_{j-s}^{-1}X_r}(\tilde{N}P_2X_{j-s}X_r^{-1}) \\ &= \mathcal{G}_{\tilde{M}X_{j-s-r}^{-1}}(\tilde{N}X_{j-s-r})\end{aligned}$$

since  $P_2X_{j-s}X_r^{-1}{}^tP_2 = X_{j-s-r}$ . If  $j = s + r$  then with Proposition 2.2(c) we have

$$\mathcal{G}_{X_{0,r}\tilde{M}P_2X_{j-s}^{-1}}(X_{0,r}\tilde{N}P_2X_{j-s}) = \mathcal{G}_{\tilde{M}}(\tilde{N}) = \mathcal{G}_M(N).$$

Suppose  $j > s + r$ . Then we modify  $G$  in our previous step to prepare to apply Proposition 5.3. Take  $G_0$  as before, and choose an integral  $(d_5 - d_8) \times r$  matrix  $W$  so that

$$\begin{pmatrix} E_0 & \\ & I_{n-d-r-d_5} \end{pmatrix} D_5 \begin{pmatrix} G_0 & \\ & I_{j-s-d_5} \end{pmatrix} \begin{pmatrix} I_{d_5-d_8} & W \\ & I_r \end{pmatrix} \equiv \begin{pmatrix} \beta''_1 & 0 \\ * & * \end{pmatrix} (q).$$

Then with

$$G_1 = \begin{pmatrix} G_0 & \\ & I_{j-s-d_5} \end{pmatrix} \begin{pmatrix} I_{d_5-d_8} & W \\ & I_r \end{pmatrix},$$

we have

$$\begin{pmatrix} E_0 & \\ & I_{n-d-r-d_5} \end{pmatrix} A_5 {}^tG_1^{-1} \equiv \begin{pmatrix} \alpha'_5 & 0 \\ * & * \end{pmatrix} (q)$$

(recall that  $\omega_2 \equiv 0 (q)$ ). Set  $G = \begin{pmatrix} G_1 & \\ & I_{n-j+s} \end{pmatrix}$ ; so  $G$  commutes with  $X_{j-s}$ . Hence we again have

$$\mathcal{G}_{X_{0,r}\tilde{M}P_2X_{j-s}^{-1}}(X_{0,r}\tilde{N}P_2X_{j-s}) = \mathcal{G}_{\tilde{M}X_{j-s-r}^{-1}}(\tilde{N}X_{j-s-r}).$$

But now, with

$$P_3 = \begin{pmatrix} I_d & 0 & 0 \\ 0 & 0 & I_{n-d-d_5+d_8} \\ 0 & I_{d_5-d_8} & 0 \end{pmatrix},$$

by Proposition 5.3 we have

$$\begin{aligned}\mathcal{G}_{\tilde{M}X_{j-s-r}^{-1}}(\tilde{N}X_{j-s-r}) &= \mathcal{G}_{P_3\tilde{M}X_{j-s-r}^{-1}}(P_3\tilde{N}X_{j-s-r}) \\ &= \left( \frac{\det \alpha'_5 \beta''_1}{q} \right) (\mathcal{G}_1(q))^{d_5-d_8} \mathcal{G}_{\tilde{M}}(\tilde{N})\end{aligned}$$



and we know by Proposition 2.2(c) that  $\mathcal{G}_{\widetilde{M}}(\widetilde{N}) = \mathcal{G}_M(N)$ . Recall that  $r = j - s - d_5 + d_8$  and  $d_5 \geq d_8$ . Hence we have  $j = s + r$  if and only if  $d_5 = d_8$ . Hence for all choices of  $s, r$ , the above computations give us

$$\mathcal{G}_{X_{s,r} M X_j^{-1}}(X_{s,r} N X_j) = q^s \left( \frac{\det \alpha'_5 \beta''_1}{q} \right) (\mathcal{G}_1(q))^{d_5 - d_8} \mathcal{G}_M(N).$$

Now, by the symmetry of  $M^t N$ , we know that  $\begin{pmatrix} \beta'_1 & \gamma'_1 \\ \delta'_7 & 0 \end{pmatrix} \begin{pmatrix} {}^t \alpha_5 & \\ & {}^t \alpha_8 \end{pmatrix}$  is symmetric modulo  $q$ , and hence

$$\left( \frac{\det \begin{pmatrix} \beta'_1 & \gamma'_1 \\ \delta'_7 & 0 \end{pmatrix} \begin{pmatrix} {}^t \alpha_5 & \\ & {}^t \alpha_8 \end{pmatrix}}{q} \right) = \left( \frac{-1}{q} \right)^{d_5 - d_8} \left( \frac{\det \alpha'_5 \beta''_1}{q} \right).$$

Also,  $\overline{\mathcal{G}}_1(q) = \mathcal{G}_{-1}(q) = \left( \frac{-1}{q} \right) \mathcal{G}_1(q)$ . Thus

$$\frac{\overline{\mathcal{G}}_M(N) \sqrt{\det N''}}{\sqrt{\det N} \overline{\mathcal{G}}_{M'}(N'')} = \frac{q^{(d_5 - d_8)/2}}{(\mathcal{G}_1(q))^{d_5 - d_8}} \left( \frac{\det \begin{pmatrix} \beta'_1 {}^t \alpha_5 & \gamma'_1 {}^t \alpha_8 \\ \delta'_7 {}^t \alpha_5 & 0 \end{pmatrix}}{q} \right).$$

To evaluate  $c_d(M_{d'}, I)$ , we also need to evaluate

$$\chi(M, N) = \chi(X_{s,r}^{-1} E M_{d'} G X_j, X_{s,r} E^t G^{-1} X_j^{-1}).$$

We note that by Lemma 6.1 [7] we can choose  $E, G \equiv I \pmod{4N/q}$ , and we can choose  $Y \equiv 0 \pmod{4N/q}$ . Thus  $M \equiv M_{d'} \pmod{4N/q}$  so

$$\chi_{4N/q}(M, N) = \chi_{4N/q}(X_{s,r}^{-1} M_{d'} X_j, X_{s,r}^{-1} X_j^{-1}).$$

As shown in the proof of Theorem 4.4 [7], with  $(M \ N)$  as above we have

$$\overline{\chi}_q(M, N) = \overline{\chi}_q \left( \det \begin{pmatrix} \beta'_1 {}^t \alpha_5 & \gamma'_1 {}^t \alpha_8 \\ \delta'_7 {}^t \alpha_5 & 0 \end{pmatrix} \det(\delta'_6 {}^t \alpha_7) \right).$$

Now we need to consider what happens when we fix  $s, r, d'$  and let  $E, G, Y$  vary so that  $X_{s,r}^{-1} E(M_{d'} G X_j - M_{d'} G Y X_j^{-1} + {}^t G^{-1} X_j^{-1}) \in SL_n(\mathbb{Z})(M_{d'} \ I)$ . As proved in Theorem 4.4 [7], there are

$$\beta_q(d, s) \beta_q(d' - d, d_5) \beta_q(n - d', j - s + d_8 - d' + d) \beta_q(d' + d - d_5, d_8) \\ \cdot q^{(d+d_5)(r+d+d_5-d') + s(n-d-d_5) + (d_4+d_8)(j-s-d_5) - d_7 d_8}$$

permissible choices for  $(E, G)$ , and for each choice of  $(E, G)$ , as  $Y$  varies over permissible choices, the matrix

$$\begin{pmatrix} \beta'_1 {}^t \alpha_5 & \gamma'_1 {}^t \alpha_8 & 0 \\ \delta'_7 {}^t \alpha_5 & 0 & 0 \\ 0 & 0 & \delta'_6 {}^t \alpha_7 \end{pmatrix}$$

varies  $q^{(j-s)(n-d+1)-d_5(j-s+d_8+1)-d_7(d_7+1)/2}$  times over all symmetric, invertible matrices modulo  $q$ . As  $\beta'_1, \gamma'_1, \delta'_6, \delta'_7$  vary as such,

$$\begin{aligned} & \sum \bar{\chi}_q \left( \det \begin{pmatrix} \beta'_1 {}^t\alpha_5 & \gamma'_1 {}^t\alpha_8 \\ \delta'_7 {}^t\alpha_5 & 0 \end{pmatrix} \det(\delta'_6 {}^t\alpha_7) \right) \left( \frac{\det \begin{pmatrix} \beta'_1 {}^t\alpha_5 & \gamma'_1 {}^t\alpha_8 \\ \delta'_7 {}^t\alpha_5 & 0 \end{pmatrix}}{q} \right) \\ &= \text{sym}_q^{\bar{\chi}'}(d_5, d_8) \text{sym}_q^{\bar{\chi}}(d_7), \end{aligned}$$

and by Lemma 2.3, this is  $\text{sym}_q^{\chi'}(d_5, d_8) \text{sym}_q^{\chi}(d_7)$ . Now combining the above results yields the theorem.  $\square$

From this theorem we can deduce a “multiplicity-one” result. To ease our description, we introduce the following.

**Definition.** Let  $\sigma, \alpha$  be multiplicative partitions of  $\mathcal{N}$ , and let  $q$  be a prime dividing  $\mathcal{N}$ . We write  $\sigma < \alpha (q)$  if  $\text{rank}_q M_\sigma < \text{rank}_q M_\alpha$ ,  $\sigma = \alpha (q)$  if  $\text{rank}_q M_\sigma = \text{rank}_q M_\alpha$ , and  $\sigma \leq \alpha (q)$  if  $\text{rank}_q M_\sigma \leq \text{rank}_q M_\alpha$ . For  $Q|\mathcal{N}$ , we write  $\sigma < \alpha (Q)$  if  $\text{rank}_q M_\sigma < \text{rank}_q M_\alpha$  for all primes  $q|Q$ ,  $\sigma = \alpha (Q)$  if  $\text{rank}_q M_\sigma = \text{rank}_q M_\alpha$  for all primes  $q|Q$ ,  $\sigma \leq \alpha (Q)$  if  $\text{rank}_q M_\sigma \leq \text{rank}_q M_\alpha$  for all primes  $q|Q$ .

**Corollary 4.4.** *Let  $\sigma$  be a partition of  $\mathcal{N}$  so that  $\mathbb{E}_\sigma \neq 0$  and let  $q$  a prime dividing  $\mathcal{N}$ ; set  $d = \text{rank}_q M_\sigma$ . For any partition  $\beta$  of  $\mathcal{N}$  with  $\beta \geq \sigma (\mathcal{N})$ , there are constants  $a_{\sigma, \beta}(\mathcal{N})$  so that  $a_{\sigma, \sigma}(\mathcal{N}) = 1$ , and with*

$$\tilde{\mathbb{E}}_\sigma = \sum_{\beta \geq \sigma(\mathcal{N})} a_{\sigma, \beta}(\mathcal{N}) \mathbb{E}_\beta,$$

we have  $\tilde{\mathbb{E}}_\sigma | T_j(q^2) = \lambda_{\sigma; j}(q^2) \tilde{\mathbb{E}}_\sigma$  where

$$\lambda_{\sigma; j}(q^2) = q^{jd} \sum_{s=0}^j q^{s(k-2d+s-j-1)} \chi_{\mathcal{N}'_0}(q^{2s}) \chi_{\mathcal{N}'_n}(q^{2(j-2)})$$

and  $\mathcal{N}'_i = \mathcal{N}_i / (q, \mathcal{N}_i)$ . For  $\sigma, \rho$  distinct multiplicative partitions of  $\mathcal{N}$ , there is some prime  $q|\mathcal{N}$  so that  $\lambda_{\sigma; n}(q^2) \neq \lambda_{\rho; n}(q^2)$ . Further,  $\tilde{\mathbb{E}}_\sigma = 0$  if and only if  $E_\sigma = 0$ , and

$$\begin{aligned} & \text{span}\{\tilde{\mathbb{E}}_\sigma : \sigma \text{ is a multiplicative partition of } \mathcal{N}\} \\ &= \text{span}\{\mathbb{E}_\sigma : \sigma \text{ is a multiplicative partition of } \mathcal{N}\}. \end{aligned}$$

*Proof.* This proof follows the lines of reasoning used to prove Corollaries 4.2 and 4.3 in [7].

First, fix a multiplicative partition  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$  of  $\mathcal{N}$  and a prime  $q|\mathcal{N}$ . Let  $d = \text{rank}_q M_\sigma$ . We temporarily use the notation of Theorem 4.3; so we write  $\sigma_d$  for  $\sigma$ , and for  $t > 0$ , we write  $\sigma_{d+t}$  for  $\rho$  where  $\rho = \sigma (\mathcal{N}/q)$  and  $\text{rank}_q M_\rho = d + t$ . Then by Theorem 4.3, we have

$$\mathbb{E}_{\sigma_d} | T_n(q^2) = \sum_{t \geq 0} A_n(d, t) \mathbb{E}_{\sigma_{d+t}};$$

if  $\mathbb{E}_{\sigma_{d+t}} = 0$  for some  $t$  then we can replace  $A_n(d, t)$  by 0 in this formula. The formula for  $A_n(d, 0)$  is a sum on  $s$  with  $0 \leq s \leq n$ , and the corresponding

summand has a term  $\beta_q(d, s)\beta_q(n-d, n-s)$ . Consequently

$$A_n(d, 0) = q^{d(k-d-1)} \overline{\chi}_{\mathcal{N}/q} \left( qX_{d,n-d}^{-1}(q), \frac{1}{q}X_{d,n-d}^{-1}(q) \right)$$

since  $\beta_q(d, s) = 0$  if  $s > d$  and  $\beta_q(n-d, n-s) = 0$  if  $s < d$ . So we can represent  $T_n(q^2)$  on  ${}^t(\mathbb{E}_{\sigma_0} \dots \mathbb{E}_{\sigma_n})$  by an upper triangular matrix, whose  $d$ th diagonal entry has absolute value  $q^{d(k-d-1)}$  when  $\mathbb{E}_{\sigma_d} \neq 0$ ; when  $\mathbb{E}_{\sigma_d} = 0$ , we can take the  $d$ th column of this matrix to be zeros. Hence we can diagonalize this matrix; correspondingly, for each  $\rho$  with  $\rho = \sigma(\mathcal{N}/q)$  and  $\mathbb{E}_\rho \neq 0$ , there are values  $a_{\rho, \alpha}(q)$  with  $a_{\rho, \rho}(q) = 1$  so that

$$\sum_{\substack{\alpha \geq \rho(q) \\ \alpha = \rho(\mathcal{N}/q)}} a_{\rho, \alpha}(q) \mathbb{E}_\alpha$$

is an eigenform for  $T_n(q^2)$  with eigenvalue  $\lambda_{\sigma; n}(q^2)$  (as defined in the statement of the corollary). Further, for  $\alpha > \rho(q)$ ,  $\alpha = \rho(\mathcal{N}/q)$ , by Proposition 2.3 and Theorem 4.3, we have  $a_{\rho, \alpha}(q) = 0$  unless  $\chi_q^2 = 1$ .

Now, for any prime  $q|\mathcal{N}$  and  $\sigma, \alpha$  multiplicative partitions of  $\mathcal{N}$  with  $\alpha \geq \sigma(q)$ , set

$$a_{\sigma, \alpha}(q) = a_{\rho, \alpha}(q)$$

where  $\rho = \sigma(q)$  and  $\rho = \alpha(\mathcal{N}/q)$ . Then for any  $Q|\mathcal{N}$  and  $\alpha \geq \sigma(Q)$ , set

$$a_{\sigma, \alpha}(Q) = \prod_{\substack{q|Q \\ q \text{ prime}}} a_{\sigma, \alpha}(q).$$

Set

$$\tilde{\mathbb{E}}_\sigma = \sum_{\alpha \geq \sigma(\mathcal{N})} a_{\sigma, \alpha}(\mathcal{N}) \mathbb{E}_\alpha.$$

So

$$\begin{aligned} \tilde{\mathbb{E}}_\sigma | T_n(q^2) &= \sum_{\substack{\beta = \sigma(q) \\ \beta \geq \sigma(\mathcal{N}/q)}} a_{\sigma, \beta}(\mathcal{N}/q) \sum_{\substack{\alpha \geq \beta(q) \\ \alpha = \beta(\mathcal{N}/q)}} a_{\beta, \alpha}(q) \mathbb{E}_\alpha | T_n(q^2) \\ &= \sum_{\substack{\beta = \sigma(q) \\ \beta \geq \sigma(\mathcal{N}/q)}} \lambda_{\beta; n}(q^2) a_{\sigma, \beta}(\mathcal{N}/q) \sum_{\substack{\alpha \geq \beta(q) \\ \alpha = \beta(\mathcal{N}/q)}} a_{\beta, \alpha}(q) \mathbb{E}_\alpha. \end{aligned}$$

We claim that for any  $\beta$  so that  $\beta = \sigma(q)$ ,  $\beta \geq \sigma(\mathcal{N}/q)$ , and  $a_{\sigma, \beta}(\mathcal{N}/q) \neq 0$ , we have  $\lambda_{\beta; n}(q^2) = \lambda_{\sigma; n}(q^2)$ . To see this, first note that since  $\beta = \sigma(q)$ , we have

$$\lambda_{\beta; n}(q^2) = q^{d(k-d-1)} \overline{\chi}_{\mathcal{N}/q} \left( qX_{d,n-d}^{-1}(q) M_\beta, \frac{1}{q}X_{d,n-d}^{-1}(q) \right).$$

Using Proposition 4.1, we see that for a prime  $q'|\mathcal{N}/q$  and  $d' = \text{rank}_{q'} M_\beta$ , we have

$$\chi_{q'} \left( qX_{d,n-d}^{-1}(q) M_\beta, \frac{1}{q}X_{d,n-d}^{-1}(q) \right) = \chi_{q'}^2(q^{d-d'})$$

and by Proposition 3.5,  $\chi_{q'}^2 = 1$  when  $q'|\mathcal{N}/\mathcal{N}_0\mathcal{N}_n$  (and necessarily  $\chi_4^2 = 1$ ). As noted above, for  $\beta > \sigma(q')$  and  $\beta \geq \sigma(\mathcal{N}/q)$ , we know that  $a_{\sigma, \beta}(q') = 0$

unless  $\chi_{q'}^2 = 1$ . Thus when  $a_{\sigma,\beta}(\mathcal{N}/q) \neq 0$ , we have  $\chi_{q'}^2 = 1$  for all primes  $q'|\mathcal{N}/q$ . Hence

$$\lambda_{\beta;n}(q^2) = q^{d(k-d-1)} = \lambda_{\sigma;n}(q^2).$$

Consequently  $\tilde{\mathbb{E}}_\sigma|T_n(q^2) = \lambda_{\sigma;n}(q^2)\tilde{\mathbb{E}}_\sigma$ .

Regardless of whether  $\chi_{\mathcal{N}/q}^2 = 1$ , we have  $|\lambda_{\sigma;n}(q^2)| = q^{d(k-d-1)}$  for every prime  $q|\mathcal{N}$  and  $d = \text{rank}_q M_\sigma$ . Hence for  $\rho \neq \sigma$  ( $\mathcal{N}$ ), there is some prime  $q|\mathcal{N}$  so that  $\text{rank}_q M_\rho \neq \text{rank}_q M_\sigma$ , and hence  $\lambda_{\rho;n}(q^2) \neq \lambda_{\sigma;n}(q^2)$ . This gives us the multiplicity-one result claimed in the statement of the corollary.

Finally, since the Hecke operators commute, we must have that  $\tilde{\mathbb{E}}_\sigma$  is an eigenform for  $T_j(q^2)$  for all primes  $q|\mathcal{N}$  and  $1 \leq j < n$ . Thus using Theorem 4.3, we must have

$$\tilde{\mathbb{E}}_\sigma|T_j(q^2) = \lambda_{\sigma;j}(q^2)\tilde{\mathbb{E}}_\sigma,$$

as claimed.  $\square$

Since the Hecke operators commute, we know that for every odd prime  $p \nmid \mathcal{N}$  we must have that  $\tilde{\mathbb{E}}_\sigma$  is an eigenform for  $T_j(p^2)$ ; below we compute the eigenvalues. These are not so attractive, so in the corollary that follows we use an alternate set of generators for the local Hecke algebra, producing much more attractive eigenvalues.

Note that in Corollary 4.4 we have only diagonalized the space of Eisenstein series corresponding to  $\Gamma_\infty \backslash \Gamma_0(4)/\Gamma_0(4\mathcal{N})$ , relative to the Hecke operators  $T_j(q^2)$  for primes  $q|\mathcal{N}$ . Following the proof of Theorem 4.3, we can see that the  $\tilde{\mathbb{E}}_\sigma$  will not all be eigenforms for  $T_j(4)$  (unless all the Eisenstein series corresponding to cusps outside  $\Gamma_\infty \backslash \Gamma_0(4)/\Gamma_0(4\mathcal{N})$  are all 0, which is certainly not the case for Siegel degree  $n = 1$ ).

**Theorem 4.5.** *Let  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$  be a multiplicative partition of  $\mathcal{N}$ , and suppose that  $\mathbb{E}_\sigma \neq 0$ . Let  $p$  be a prime not dividing  $4\mathcal{N}$ , and take  $j$  so that  $1 \leq j \leq n$ . Then  $\mathbb{E}_\sigma|T_j(p^2) = \lambda_j(p^2)\mathbb{E}_\sigma$  where*

$$\begin{aligned} \lambda_{\sigma;j}(p^2) = & \beta_p(n, j) \sum_{r+s \leq j} p^{k(j-r+s)/2 - (j-r)(n+1)} \chi(p^{j-r+s}) \chi_{\mathcal{N}_n}(p^{2(r-s)}) \\ & \cdot \beta_p(j, r) \beta_p(j-r, s) \left( \frac{\mathcal{G}_1(p)}{\sqrt{p}} \right)^{j-r-s} \text{sym}_p^\psi(j-r-s) \end{aligned}$$

where the sum is over all non-negative integers  $r, s$  with  $r + s \leq j$ , and  $\psi(*) = \left( \frac{*}{p} \right)$ . Further,  $\tilde{\mathbb{E}}_\sigma|T_j(p^2) = \lambda_{\sigma;j}(p^2)\tilde{\mathbb{E}}_\sigma$ .

*Proof.* To a large extent we follow the reasoning of Theorem 5.4 [7].

For any  $n_0, n_2 \in \mathbb{Z}_{\geq 0}$  with  $n_0 + n_2 \leq j$ ,  $G \in SL_n(\mathbb{Z})$ ,  $Y \in \mathcal{Y}_{n_0, n_2}$ , and  $SL_n(\mathbb{Z})(M \ N) \in \tilde{SL}_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4\mathcal{N})$ , we adjust the representative  $(M, N)$  and choose  $r, s \in \mathbb{Z}_{\geq 0}$  so that

$$(M' \ N'') = X_{r,s}^{-1}(MX_{n_0, n_2}^{-1}G^{-1} \ NX_{n_0, n_2}^t G)$$

is an integral coprime pair (which is automatically symmetric). Note that  $M' \equiv 0 \pmod{4}$ , and for all primes  $q|\mathcal{N}$ , we have  $\text{rank}_q M' = \text{rank}_q M_\sigma$ . It follows from Proposition 3.5 that  $(M' \ N'') \in SL_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4\mathcal{N})$ . Then as in

the proof of Theorem 4.3, we have

$$\begin{aligned} S_{M,N}(X_{n_0,n_2}^{-1}G^{-1}(\tau + GY^tG)^tG^{-1}X_{n_0,n_2}^{-1}) \\ = p^{(r-s-n_0+n_2)/2}S_{M',N''}(\tau + GY^tG) \\ = p^{(r-s-n_0+n_2)/2}\frac{\sqrt{\det N''}}{\overline{\mathcal{G}}_{M'}(N'')}\frac{\overline{\mathcal{G}}_{M'}(N')}{\sqrt{\det N'}}S_{M',N'}(\tau) \end{aligned}$$

where  $N' = M'Y + N''$  (and  $(M' \ N') \in SL_n(\mathbb{Z})(M_\sigma \ I)\Gamma_0(4N)$ ). Setting  $\varepsilon = \left(\frac{-1}{p}\right)$ , we have

$$\mathbb{E}_\sigma|T_j(p^2) = \sum_{(M' \ N')} c_\sigma(M', N')\overline{\chi}(M', N') \left(\frac{\overline{\mathcal{G}}_{M'}(N')}{\sqrt{\det N'}}S_{M',N'}(\tau)\right)^{-k}$$

with

$$\begin{aligned} c_\sigma(M', N') &= p^{j(k/2-n-1)}\chi(M', N') \sum \chi(p^{j-n_0+n_2})\overline{\chi}(M, N) \\ &\quad \cdot \left(\frac{\overline{\mathcal{G}}_M(N)}{\sqrt{\det N}}\frac{\sqrt{\det N''}}{\overline{\mathcal{G}}_{M'}(N'')}\right)^{-k} p^{k(s-r)/2} \\ &\quad \cdot \varepsilon^{(k+1)(j-n_0-n_2)/2} \left(\frac{\det Y_1}{p}\right) (p^{-1/2}\mathcal{G}_1(p))^{j-n_0-n_2}, \end{aligned}$$

where the sum is over all  $r, s, n_0, n_2 \in \mathbb{Z}_{\geq 0}$  with  $n_0 + n_2 \leq j$ ,  $G, Y$  as in Proposition 4.1(b), and  $E \in \mathcal{K}_{s,r} \setminus SL_n(\mathbb{Z})$  so that

$$\begin{aligned} (M \ N) &= X_{r,s}E(M'GX_{n_0,n_2}^{-1} - M'GYX_{n_0,n_2}^{-1} + N'^tG^{-1}X_{n_0,n_2}^{-1}) \\ &\in SL_n(\mathbb{Z})(M_{\sigma'} \ I). \end{aligned}$$

Now fix a partition  $\sigma'$  of  $N$ . As noted in §5 of [7], we can choose the representative  $M_{\sigma'}$  to be divisible by  $p^3$ ; then with  $(M \ N)$  as above, we have that  $p|M$  and  $N$  is invertible modulo  $p$ . We also have that  $N'' = X_{r,s}^{-1}NX_{n_0,n_2}^{-1}G$  is invertible modulo  $p$ ; consequently (as proved in Theorem 5.4 [7]), we must have  $r = n_0$  and  $s = n_2$ . From this we find that

$$N = \begin{pmatrix} N_1 & pN_2 & p^2N_3 \\ N_4 & N_5 & pN_6 \\ N_7 & N_8 & N_9 \end{pmatrix}$$

with  $N_1$   $r \times r$ ,  $N_9$   $s \times s$ , and  $N_1, N_5, N_9$  invertible modulo  $p$ . Hence  $X_r^{-1}NX_r$  is also integral and invertible modulo  $p$ . Set

$$P = \begin{pmatrix} 0 & 0 & I_s \\ 0 & I & 0 \\ I_r & 0 & 0 \end{pmatrix} \quad (n \times n).$$

Then  $X_r^{-1}N^tPX_{0,r}^{-1} = X_r^{-1}NX_rP$  is invertible modulo  $p$ , and using Proposition 2.2(c) and Proposition 5.2, we have

$$\begin{aligned}
\mathcal{G}_{M_{\sigma'}}(N'') &= \mathcal{G}_{X_{r,s}^{-1} M X_{r,s}^{-1}}(X_{r,s}^{-1} N X_{r,s}) \\
&= \mathcal{G}_{X_{0,s}^{-1}(X_r^{-1} M {}^t P X_{0,r}) X_s}(X_{0,s}^{-1}(X_r^{-1} N {}^t P X_{0,r}^{-1}) X_s^{-1}) \\
&= \mathcal{G}_{X_r^{-1} M {}^t P X_{0,r}}(X_r^{-1} N {}^t P X_{0,r}^{-1}) \\
&= \mathcal{G}_{X_{0,r} P M X_r^{-1}}(X_{0,r} P N X_r) \\
&= \mathcal{G}_M(N).
\end{aligned}$$

As shown in the proof of Theorem 4.5 [7], given  $r, s$ , for all choices of  $Y$  and for

$$p^{rs} \beta_p(n, j) \beta_p(j, r) \beta_p(j - r, s)$$

choices of  $(E, G)$  we have

$$\begin{aligned}
(M \ N) &= X_{r,s} E(M_{\sigma'} G X_{r,s} - M_{\sigma'} G Y X_{r,s}^{-1} + {}^t G^{-1} X_{r,s}^{-1}) \\
&\in SL_n(\mathbb{Z})(M_{\sigma'} \ I) \Gamma_0(4N).
\end{aligned}$$

Writing  $Y$  as in Proposition 4.2, we have

$$\sum_Y \left( \frac{\det Y_1}{p} \right) = p^{r(n-s+1)} \text{sym}_p^\psi(j - r - s)$$

where  $\psi_p(*) = \left( \frac{*}{p} \right)$ . Note that  $\text{sym}_p^\psi(j - r - s) = 0$  when  $j - r - s$  is odd, and when  $j - r - s$  is even we have  $\varepsilon^{(k+1)(j-r-s)/2} = 1$ . Also, we can always choose  $E, G \equiv I \pmod{p}$  and  $Y \equiv 0 \pmod{p}$ ; hence  $\overline{\chi}(M, N) = \overline{\chi}(X_{r,s} M_{\sigma'} X_{r,s}, I)$ , and by Proposition 4.1, if  $\mathbb{E}_{\sigma'} \neq 0$ , we have  $\overline{\chi}(M, N) = \chi_{\mathcal{N}_n}(p^{2(r-s)})$ . Combining these computations yields the value of  $\lambda_{\sigma;j}(p^2)$ .

Since the Hecke operators commute, by our multiplicity one result (Corollary 4.4),  $\tilde{\mathbb{E}}_\sigma = \sum_{\alpha \geq \sigma} a_{\sigma,\alpha} \mathbb{E}_\alpha$  must be a  $T_j(p^2)$  eigenform. Since each  $\mathbb{E}_\alpha$  is a  $T_j(p^2)$  eigenform, we must have  $\lambda_{\alpha;j}(p^2) = \lambda_{\sigma;j}(p^2)$  whenever  $a_{\sigma,\alpha} \neq 0$  (which can be corroborated by direct computation), so we have  $\tilde{\mathbb{E}}_\sigma|T_j(p^2) = \lambda_{\sigma;j}(p^2) \tilde{\mathbb{E}}_\sigma$ .  $\square$

**Corollary 4.6.** *Let  $p$  be a prime not dividing  $4N$ , and set  $\varepsilon = \left( \frac{-1}{p} \right)$ . Set*

$$\tilde{T}_j(p^2) = \sum_{\ell=0}^j \chi(p^{j-\ell}) \varepsilon^{(k+1)(j-\ell)/2} p^{(j-\ell)(k/2-n-1/2)} \beta_p(n-\ell, j-\ell) T_\ell(p^2)$$

and

$$T'_j(p^2) = \sum_{i=0}^j (-1)^i p^{i(i-1)/2} \beta_p(n-j+i, i) \chi_{\mathcal{N}_n}(p^{2i}) \tilde{T}_{j-i}(p^2).$$

With  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$  a multiplicative partition of  $\mathcal{N}$ , we have

$$\mathbb{E}_\sigma|T'_j(p^2) = \lambda'_j(p^2) \mathbb{E}_\sigma \text{ and } \tilde{\mathbb{E}}_\sigma|T'_j(p^2) = \lambda'_{\sigma;j}(p^2) \tilde{\mathbb{E}}_\sigma$$

where

$$\lambda'_{\sigma;j}(p^2) = \beta_p(n, j) p^{j(k/2-n-1/2)+j(j-1)/2} \chi'(p^j) \prod_{i=1}^j (\chi' \overline{\chi}_{\mathcal{N}_n}^2(p) p^{(k+1)/2-i} + 1),$$

and  $\chi'(p^s) = \chi(p^s) \varepsilon^{s(k+1)/2}$ .

*Proof.* To take advantage of a result proved in [6], we set

$$\tilde{\mathcal{G}}(\langle 0 \rangle^\ell) = p^{-\ell} (\mathcal{G}_1(p))^\ell \text{sym}_p^\psi(\ell)$$

where  $\psi(*) = \left(\frac{*}{p}\right)$ . Then (with  $t = \ell - r - s$ ) we have  $\mathbb{E}_\sigma[\tilde{T}_j(p^2)] = \tilde{\lambda}_j(p^2) \mathbb{E}_\sigma$  where

$$\begin{aligned} \tilde{\lambda}_{\sigma;j}(p^2) &= \sum_{0 \leq \ell \leq j} \chi(p^{j-\ell})_{\varepsilon^{(k+1)(j-\ell)/2}} p^{(j-\ell)(k/2-n-1/2)} \\ &\quad \cdot \beta(n-\ell, j-\ell) \lambda_{\sigma;\ell}(p^2) \\ &= \sum \chi(p^{j-r+s}) \chi_{\mathcal{N}_n}(p^{2(r-s)})_{\varepsilon^{(k+1)(j-r-s)/2}} \\ &\quad \cdot p^{(j-r)(k/2-n-1/2)+s(k-1)/2} \\ &\quad \cdot \beta(n-\ell, j-\ell) \beta(n, \ell) \beta(\ell, r) \beta(\ell-r, s) \tilde{\mathcal{G}}(\langle 0 \rangle^{\ell-r-s}) \end{aligned}$$

where  $0 \leq \ell \leq j$  and  $0 \leq r+s \leq \ell$ , or equivalently,  $0 \leq r+s \leq j$  and  $r+s \leq \ell \leq j$ . We have

$$\begin{aligned} &\beta(n-\ell, j-\ell) \beta(n, \ell) \beta(\ell, r) \beta(\ell-r, s) \frac{\mu(j, \ell)}{\mu(j, \ell)} \\ &= \beta(n, j) \beta(j, r) \beta(j-r, s) \beta(j-r-s, \ell-r-s). \end{aligned}$$

Now we make the change of variables  $\ell \mapsto \ell - r - s$ . So

$$\begin{aligned} \tilde{\lambda}_{\sigma;j}(p^2) &= \sum \chi(p^{j-r+s}) \chi_{\mathcal{N}_n}(p^{2(r-s)})_{\varepsilon^{(k+1)(j-r-s)/2}} \\ &\quad \cdot p^{(j-r)(k/2-n-1/2)+s(k-1)/2} \\ &\quad \cdot \beta(n, j) \beta(j, r) \beta(j-r, s) \beta(j-r-s, \ell) \tilde{\mathcal{G}}(\langle 0 \rangle^\ell) \end{aligned}$$

where  $0 \leq r+s \leq j$ ,  $0 \leq \ell \leq j-r-s$ . Taking  $W = \langle 0 \rangle^{j-r-s}$  in Lemma 3.2(b) [6], we have that

$$\sum_{\ell=0}^{j-r-s} \beta(j-r-s, \ell) \tilde{\mathcal{G}}(\langle 0 \rangle^\ell) = p^{(j-r-s)(j-r-s-1)/2}$$

(as the form  $W \perp \langle 2 \rangle$  primitively represents  $\langle 0 \rangle^{j-r-s}$  only once). Hence

$$\begin{aligned} \lambda'_{\sigma;j}(p^2) &= \sum (-1)^i p^{i(i-1)/2} \chi_{\mathcal{N}_n}(p^{2(i+r-s)}) \chi'(p^{j-i-r+s}) \\ &\quad \cdot p^{(j-i-r)(k/2-n-1/2)+s(k-1)/2+(j-i-r-s)(j-i-r-s-1)/2} \\ &\quad \cdot \beta(n-j+i, i) \beta(n, j-i) \beta(j-i, r) \beta(j-i-r, s) \end{aligned}$$

where  $0 \leq i \leq j$  and  $0 \leq r+s \leq j-i$ , or equivalently,  $0 \leq i \leq j$ ,  $0 \leq r \leq j-i$ ,  $0 \leq s \leq j-i-r$ . Making the change of variables  $r \mapsto j-i-r$ , we get  $\lambda'_j(p^2)$  as a sum over  $0 \leq i \leq j$ ,  $0 \leq r \leq j-i$ ,  $0 \leq s \leq r$ , or equivalently,  $0 \leq r \leq j$ ,  $0 \leq i \leq j-r$ ,  $0 \leq s \leq r$ . We have  $\beta(j-i, j-i-r) = \beta(j-i, r)$  and

$$\beta(n-j+i, i) \beta(n, j-i) \beta(j-i, r) \frac{\mu(j, i)}{\mu(j, i)} = \beta(n, j) \beta(j, r) \beta(j-r, i).$$

Using that  $\beta(m, r) = p^r \beta(m-1, r) + \beta(m-1, r-1)$ , we find that

$$\sum_{i=0}^{j-r} (-1)^i p^{i(i-1)/2} \beta(j-r, i) = \begin{cases} 1 & \text{if } r = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\lambda'_{\sigma;j}(p^2) = \beta(n, j) \chi'(p^j) p^{j(k/2-n-1/2)+j(j-1)/2} A(j, (k+1)/2-j)$$

where

$$A(j, y) = \sum_{s=0}^j \phi(p^s) p^{ys+s(s-1)/2} \beta(j, s)$$

and  $\phi(p^s) = \chi(p^s) \varepsilon^{(k+1)s/2} \bar{\chi}_{\mathcal{N}_n}(p^{2s})$ . Again using the relation  $\beta(j, s) = p^s \beta(j-1, s) + \beta(j-1, s-1)$ , we find that

$$A(j, y) = (\phi(p) p^y + 1) A(j-1, y+1) = \prod_{i=0}^{j-1} (\chi'(p) \bar{\chi}_{\mathcal{N}_n}(p^2) p^{y+i} + 1).$$

Taking  $y = (k+1)/2 - j$  shows that  $\mathbb{E}_\sigma |T'_j(p^2) = \lambda'_{\sigma;j}(p^2) \mathbb{E}_\sigma$ .

Now recall that  $\tilde{\mathbb{E}}_\sigma = \sum_{\beta \geq \sigma(\mathcal{N})} a_{\sigma,\beta}(\mathcal{N}) \mathbb{E}_\beta$ . Suppose that  $\beta \geq \sigma(\mathcal{N})$  so that  $a_{\sigma,\beta}(\mathcal{N}) \neq 0$ . Write  $\sigma = (\mathcal{N}_0, \dots, \mathcal{N}_n)$ ,  $\beta = (\mathcal{N}'_0, \dots, \mathcal{N}'_n)$ . For any prime  $q | \mathcal{N}_n$ , we know that  $\text{rank}_q M_\beta \geq \text{rank}_q M_\sigma = n$ . Consequently  $\mathcal{N}_n | \mathcal{N}'_n$ . Now suppose that  $q$  is a prime so that  $q | \mathcal{N}'_n$  but  $q \nmid \mathcal{N}_n$ . Thus  $n = \text{rank}_q M_\beta > \text{rank}_q M_\sigma$ , as thus as discussed in the proof of Corollary 4.4, we have  $\chi_q^2 = 1$ . Hence  $\chi_{\mathcal{N}_n}^2 = \chi_{\mathcal{N}'_n}^2$ . Therefore  $\lambda'_{\beta;j}(p^2) = \lambda'_{\sigma;j}(p^2)$  for all  $\beta \geq \sigma(\sigma)$ . Hence  $\tilde{\mathbb{E}}_\sigma |T'_j(p^2) = \lambda'_{\sigma;j}(p^2) \tilde{\mathbb{E}}_\sigma$ .  $\square$

## 5. RELATIONS ON GAUSS SUMS

To prove the following identities, we frequently use that with  $n \times n$  matrices  $A, B$ , we have  $\text{Tr}(AB) = \text{Tr}(BA)$ .

**Proposition 5.1.** *Suppose  $(M, N)$ ,  $(X_s M X_s^{-1}, X_s N X_s)$  are coprime symmetric pairs. Then*

$$\mathcal{G}_{X_s M X_s^{-1}}(X_s N X_s) = q^s \cdot \mathcal{G}_M(N).$$

*Proof.* We let  $U_0$  vary over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} N X_s$ ,  $U_1$  over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} X_s$ . Then  $U_1 N X_s$  varies over  $\mathbb{Z}^{1,n} N X_s / \mathbb{Z}^{1,n} X_s N X_s$ ; hence  $U = U_0 + U_1 N X_s$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} X_s N X_s$  and, recalling that  $M^t N = N^t M$ , we have

$$e\{2^t U U X_s^{-1} N^{-1} M X_s^{-1}\} = e\{2^t U_0 U_0 X_s^{-1} N^{-1} M X_s^{-1}\} \cdot e\{4^t U_0 U_1 M X_s^{-1}\}.$$

For fixed  $U_0$ ,

$$\sum_{U_1} e\{4^t U_0 U_1 M X_s^{-1}\} = \sum_{U_1} \{4 U_1 M X_s^{-1} {}^t U_0\}$$

is a character sum, so the sum is 0 unless  $M X_s^{-1} {}^t U_0$  is integral. Since  $X_s M X_s^{-1}$  is integral,  $M = \begin{pmatrix} A_1 & A_2 \\ q A_3 & A_4 \end{pmatrix}$  with  $A_1$  of size  $s \times s$ . Since

$$(X_s M X_s^{-1}, X_s N X_s) = 1,$$



we must have that  $A_1$  is invertible modulo  $q$ .  $MX_s^{-1}{}^tU_0$  is integral if and only if  $U_0 \in \mathbb{Z}^{1,n}X_s$ , from which the proposition follows.  $\square$

**Proposition 5.2.** *Suppose  $(M \ N)$  and  $(X_{0,r}MX_r^{-1} \ X_{0,r}NX_r)$  are integral, coprime symmetric pairs. Then*

$$\mathcal{G}_{X_{0,r}MX_r^{-1}}(X_{0,r}NX_r) = \mathcal{G}_M(N).$$

*Proof.* First, note that since  $(X_{0,r}MX_r^{-1} \ X_{0,r}NX_r)$  is an integral coprime pair, we have  $M = \begin{pmatrix} qA_1 & A_2 \\ q^2A_3 & qA_4 \end{pmatrix}$ ,  $N = \begin{pmatrix} B_1 & B_2 \\ B_3 & qB_4 \end{pmatrix}$  with  $A_3, B_3$  of size  $r \times r$  and  $B_3$  invertible modulo  $q$ .

We obtain the desired identity by evaluating in two ways the sum

$$\sum_{U \in \mathbb{Q}^{1,n}/\mathbb{Z}^{1,n}NX_r} e\{2^t U U X_r^{-1} N^{-1} M X_r^{-1}\}.$$

We now show that as  $U_0$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}N$  and  $U_1$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}X_{0,r}^{-1}$ ,  $U_0X_r + U_1X_{0,r}NX_r$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}NX_r$ . Define the additive homomorphism  $\psi : \mathbb{Z}^{1,n} \times \mathbb{Z}^{1,n} \rightarrow \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}NX_r$  by

$$\psi((U_0, U_1)) = U_0X_r + U_1X_{0,r}NX_r + \mathbb{Z}^{1,n}NX_r.$$

Suppose  $(U_0, U_1) \in \ker \psi$ . Thus

$$U_0 + U_1X_{0,r}N \in \mathbb{Z}^{1,n}N \subseteq \mathbb{Z}^{1,n}.$$

Writing  $U_1 = (W_1 \ W'_1)$  where  $W'_1$  is  $1 \times r$ , we must have  $W'_1 \equiv 0 \pmod{q}$  since  $B_3$  is invertible modulo  $q$  and  $U_1X_{0,r}N$  is integral. Hence  $U_1X_{0,r}$  is integral, and thus  $U_0 \in \mathbb{Z}^{1,n}N$ . Thus

$$\ker \psi = \mathbb{Z}^{1,n}N \times \mathbb{Z}^{1,n}X_{0,r}^{-1}.$$

Since

$$|\psi(\mathbb{Z}^{1,n} \times \mathbb{Z}^{1,n})/\ker \psi| = q^r \det N = |\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}NX_r|,$$

$\psi$  is an isomorphism.

Thus with  $U_0$  varying over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}N$  and  $U_1$  over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}X_{0,r}^{-1}$ , we have

$$\begin{aligned} & \sum_{U \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}NX_r} e\{2^2 U U X_r^{-1} N^{-1} M X_r^{-1}\} \\ &= \sum_{U_0, U_1} e\{2^t (U_0X_r + U_1X_{0,r}NX_r)(U_0X_r + U_1X_{0,r}NX_r)X_r^{-1}N^{-1}MX_r^{-1}\} \\ &= \sum_{U_0, U_1} e\{2^t U_0U_0N^{-1}M\} e\{4^t U_0U_1X_{0,r}M\} e\{2^t NX_{0,r}{}^tU_1U_1X_{0,r}M\} \\ &= q^r \mathcal{G}_M(N) \end{aligned}$$

since  $X_{0,r}M$ ,  $MX_{0,r}$ , and  $X_{0,r}M^tNX_{0,r}$  are integral.

On the other hand, as  $V_0$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}X_{0,r}NX_r$  and  $V_1$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n}X_{0,r}^{-1}$  and hence  $V_1X_{0,r}NX_r$  varies over  $\mathbb{Z}^{1,n}X_{0,r}NX_r/\mathbb{Z}^{1,n}NX_r$ . So

$V_0 + V_1 X_{0,r} N X_r$  varies over  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} N X_r$ . Thus

$$\begin{aligned} & \sum_{U \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} N X_r} e\{2^t U U X_r^{-1} N^{-1} M_r^{-1}\} \\ &= \sum_{V_0, V_1} e\{2^t V_0 V_0 X_r^{-1} N^{-1} M X_r^{-1}\} e\{4^t V_0 V_1 X_{0,r} M X_r^{-1}\} \\ & \quad e\{2^t V_1 V_1 \cdot X_{0,r} M^t N X_{0,r}\} \\ &= \sum_{V_0, V_1} e\{2^t V_0 V_0 X_r^{-1} N^{-1} M X_r^{-1}\} \end{aligned}$$

since  $X_{0,r} M^t N X_{0,r}$  is integral. Thus

$$\sum_{U \in \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} N X_r} e\{2^2 U U X_r^{-1} N^{-1} M_r^{-1}\} = q^r \mathcal{G}_{X_{0,r} M X_r^{-1}}(X_{0,r} N X_r).$$

This proves the proposition.  $\square$

**Proposition 5.3.** *Suppose that  $(M \ N)$  is a coprime symmetric pair so that*

$$M = \begin{pmatrix} qB_1 & B_2 \\ qB_3 & qB_4 \end{pmatrix}, \quad N = \begin{pmatrix} C_1 & C_2 \\ C_3 & qC_4 \end{pmatrix}$$

where  $B_3, C_3$  are  $\ell \times \ell$  and invertible modulo  $q$ . Then  $\text{rank}_q(B_2 \ C_2) = n - \ell$ ,  $(M X_\ell^{-1} \ N X_\ell)$  is a coprime symmetric pair, and

$$\mathcal{G}_{M X_\ell^{-1}}(N X_\ell) = \left( \frac{\det B_3 C_3}{q} \right) (\mathcal{G}_1(q))^\ell \mathcal{G}_M(N).$$

*Proof.* Since  $C_3$  is invertible modulo  $q$ , we have

$$n = \text{rank}_q \begin{pmatrix} B_2 & 0 & C_2 \\ 0 & C_3 & 0 \end{pmatrix} = \text{rank}_q \begin{pmatrix} B_2 & 0 & C_2 \\ 0 & C_3 & 0 \end{pmatrix},$$

hence  $\text{rank}_q(B_2 \ C_2) = n - \ell$ . Also,  $\text{rank}_q(M X_\ell^{-1} \ N X_\ell) = \text{rank}_q \begin{pmatrix} 0 & B_2 & C_2 \\ B_3 & 0 & 0 \end{pmatrix} = n$ , so  $(M X_\ell^{-1} \ N X_\ell)$  is a coprime symmetric pair. We know that  $X_{0,\ell} N X_\ell$  is integral, so we define the additive homomorphism  $\psi : \mathbb{Z}^{1,n} \times \mathbb{Z}^{1,n} \rightarrow \mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} N X_\ell$  by

$$\psi((U_0, U_1)) = U_0 X_\ell + U_1 X_{0,\ell} N X_\ell + \mathbb{Z}^{1,n} N X_\ell.$$

Then just as proved in Proposition 5.2,  $\psi$  is surjective with kernel  $\mathbb{Z}^{1,n} N \times \mathbb{Z}^{1,n} X_{0,\ell} N X_\ell$ . Thus, as  $X_{0,\ell} M$  is integral, we find that

$$\mathcal{G}_{M X_\ell^{-1}}(N X_\ell) = \mathcal{G}_M(N) \mathcal{G}_{M^t N X_{0,\ell}}(X_{0,\ell}^{-1}).$$

To evaluate  $\mathcal{G}_{M^t N X_{0,\ell}}(X_{0,\ell}^{-1})$ , we first note that  $\{(0 \ V) : V \in \mathbb{Z}^{1,\ell}/q\mathbb{Z}^{1,\ell}\}$  is a set of representatives for  $\mathbb{Z}^{1,n}/\mathbb{Z}^{1,n} X_{0,\ell}^{-1}$ . Thus

$$\mathcal{G}_{M^t N X_{0,\ell}}(X_{0,\ell}^{-1}) = \sum_{V \in \mathbb{Z}^{1,\ell}/q\mathbb{Z}^{1,\ell}} e\{2^t V V B_3^t C_3/q\}.$$

Since  $q \neq 2$  and  $B_3^t C_3$  is symmetric and invertible modulo  $q$ , by section 2.8 [2], there is some  $G \in SL_\ell(\mathbb{Z})$  and  $w_1, \dots, w_\ell \in \mathbb{Z}$  so that

$$G B_3^t C_3^t G \simeq \langle w_1, w_2, \dots, w_\ell \rangle$$

where  $q \nmid w_1 w_2 \cdots w_\ell$ . Since  $\mathbb{Z}^{1,\ell} G / q \mathbb{Z}^{1,\ell} G = \mathbb{Z}^{1,\ell} / q \mathbb{Z}^{1,\ell}$ , replacing  $V$  by  $VG$  gives us

$$\begin{aligned} \mathcal{G}_{M^t N X_{0,\ell}}(X_{0,\ell}^{-1}) &= \sum_{v_1, \dots, v_\ell (q)} e\{2(v_1^2 w_1 + v_2^2 w_2 + \cdots + v_\ell^2 w_\ell)/q\} \\ &= \left( \frac{w_1 w_2 \cdots w_\ell}{q} \right) (\mathcal{G}_1(q))^\ell \\ &= \left( \frac{\det B_3 C_3}{q} \right) (\mathcal{G}_1(q))^\ell, \end{aligned}$$

proving the proposition.  $\square$

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